Identification and estimation of sequential English auctions*

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Abstract

We consider two-stage sequential English auctions for the sale of two units of an homogenous good in the independent private values paradigm and with buyers having multi-unit demands. We first investigate identification from the transaction prices, revisiting thus the claims made by Brendstrup (2007) and Brendstrup and Paarsch (2006). We show that identification fails without any assumption on the bidding behavior at the first stage. More surprisingly, we show that identification may also fail when equilibrium behavior is assumed, but also that it is restored either if there are at least three potential buyers or if the two units for sale are not perfectly identical. Second, an estimation procedure that uses the transaction price at both stages is developed and supported by Monte Carlo experiments. We implement our methodology to tobacco auctions held in southeastern United States. We are then able to run various counterfactuals: we estimate the optimal reserve price under different scenarios concerning the seller’s valuation and analyze the impact of bundling.

Keywords: Sequential auctions, multi-object auctions, optimal reserve price, bundling, nonparametric identification, nonparametric estimation, tobacco auctions.

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1 Introduction

The derivation of an equilibrium in sequential auctions with multi-unit demands and any number of stages is known to be untractable without very stringent conditions. The most general treatment is from Gale and Stegeman (2001) where the authors completely characterize a unique equilibrium allocation in a complete information framework with two buyers. Incomplete information adds new caveats. First, two buyers may become asymmetrically informed about the valuations of a third opponent. That is the reason why the information disclosure rules of previous bids are crucial in those game-theoretical analysis, even for sequential auctions with unit-demand as in Milgrom and Weber (2000). Second, “natural” models with multi-unit demands should involve multi-dimensional signals which typically raise tractability issues. In particular, Katzman’s (1999) analysis of two-stage sequential second-price auctions (where the second stage is thus dominant strategy solvable) with multi-unit demands and under incomplete information is limited to symmetric equilibria with so-called ‘separable’ bid functions where each buyer bases his first-stage bid either solely on his high valuation or solely on his low valuation. Furthermore, endogenous valuations may arise if there are more than three buyers (or with two buyers and binding reserve prices): in the earliest stages of the auction, the valuation of a buyer may depend on the identity of the winner he anticipates if he loses the auction, which opens the door for strategic nonparticipation à la Jehiel and Moldovanu (1996) and/or multiple equilibria. Finally, risk-aversion is another source of difficulty since it may prevent the existence of equilibria in pure strategies as established by McAfee and Vincent (1993) under unit-demand.

The lack of established theoretical benchmarks for sequential auctions seems to leave little room for a structural approach. However, for the sequential sale of identical units of a good through any number of stages and in the independent private values (IPV) paradigm with a special multi-unit demand structure, a model that is labeled next as ID-sampling schemes and that has been introduced by Katzman (1999), Brendstrup (2007) and Brendstrup and Paarsch (2006), henceforth B&BP, propose a methodology that relies solely on the fact that at the last stage of the game and if this last stage is an English (or equivalently a second-price) auction, then it is a weakly dominant strategy for each buyer to bid up to his marginal valu-
ation for winning the last unit given the previous units he has won in earlier stages.\(^1\) More precisely, the unique assumption B&BP impose on their sequential auction model is that the transaction (or winning) price at the last stage corresponds to the second-highest marginal valuation for winning the last unit, a mild equilibrium behavior assumption that, e.g., does not depend on whether buyers are risk-averse or not. Then B&BP claim that the joint distribution of valuations can be identified from the distribution of the last stage winning price without any further assumption, i.e. in particular on the bidding behavior in the earlier stages. Brendstrup (2007) proposes a related nonparametric estimation procedure when buyers are symmetric while Brendstrup and Paarsch (2006) propose a semi-nonparametric estimation procedure in the more general case with possibly asymmetric buyers.\(^2\) Those papers correctly recognize that, even if bidders are symmetric ex ante, the outcomes of earlier auctions lead to endogenous asymmetry among bidders in the last auction. Nevertheless, their derivations do not account for a selection bias: it does not fully handle all the informational content embraced by the number of units obtained by the bidders in the first stages, in particular, the one resulting from the strategic nature of the previous interactions between bidders. In other words, for a given distribution of the latent valuations, the distribution of the winning price at the last stage does not solely depend on the number of units assigned to the various buyers in earlier stages as claimed by B&BP but also crucially on the way buyers have bid in earlier stages.\(^3\) B&BP’s identification and estimation procedures are valid under a bidding heuristic where buyers bid randomly, i.e. independently of their private values, in the first stage, a bidding heuristic that is not an equilibrium. Without any specific assumption on the bidding heuristic, we show actually that the model considered in B&BP is not identified. Furthermore, with only two potential buyers and when the two units are perfectly identical, we show that their model is not identified even if equilibrium behavior is assumed for the whole sequential auction game. The lack of identification with two buyers under equilibrium behavior comes from

\(^1\)By imposing a specific demand-generation scheme for buyers’ valuations that guarantees a kind of stationarity, Donald et al. (2006) are able to exploit the winning bid from all stages in a structural way in sequential English auctions.

\(^2\)More precisely, Brendstrup and Paarsch (2006) consider a model with possibly asymmetric buyers and their identification claim is then obtained from the distribution of the winning price at the last stage and also the identities of the winners in all stages.

\(^3\)The same ‘selection bias’ also arises in Brendstrup’s (2006) analysis of sequential English auctions with heterogeneous objects with synergies.
the multiplicity of equilibria, a phenomenon already established by Katzman (1999)
and which suggests that little can be done without solving the auction game.

In order to develop a structural approach, we are relying then extensively on
Lamy (2012) which characterizes the set of (symmetric pure-strategy) equilibria in
two-stage sequential second-price auctions with multi-unit demands and establishes
equilibrium uniqueness under mild conditions. Lamy (2012) does not solely drop
the ad hoc ‘separable’ bid functions restriction imposed by Katzman (1999) but
also consider a significantly more general framework than the one considered in
Katzman (1999) and B&BP. In particular, Lamy (2012) covers the case where the
two units for sale are not perfectly identical though still coming from an homogenous
good. This is crucial in our application where the two lots of tobacco may be of
different weight.\(^4\) In order to be able to apply the results in Lamy (2012), our
subsequent analysis is then confined to the case of two-stage sequential auctions and
to symmetric buyers. While it reduces significantly the scope of the analysis with
respect to B&BP’s setting, we emphasize that we consider a much more general
framework in other respects. When there are at least three potential buyers or when
the second lot is smaller than the first lot, the equilibrium is actually unique and
the equilibrium bid function in the first stage depends solely on the high valuation,
i.e. the marginal valuation for winning only one unit. With this known form of
equilibrium, we can easily control for the aforementioned selection bias and then
revisit B&BP’s analysis. In B&BP’s setting, i.e. under ID-sampling schemes where
the high and the low valuations of any given bidder correspond to two independent
draws from a common distribution, we establish an analog (but different) one-to-one
mapping between the underlying distribution that generates buyers’ valuation and
the distribution of last stage’s transaction price. We go also one step further by
considering more general valuation sampling schemes: as in Lamy (2012), we also
consider models where valuations are drawn independently from a general bivariate
distribution, i.e. the most general form of substitutes preferences that covers not
solely Katzman (1999) and B&BP’s framework but also unit-demand or flat-demand
as special extreme cases, and we establish nonparametric identification from the joint
distribution of the winning prices at both stages.

Our interest in identifying the distribution of buyers’ valuations lies mainly in the
\(^4\)On the contrary, in the fish auctions analyzed by B&BP, the fish are packed into units of exactly
the same weight.
fact that it is the primitive for the computation of revenue maximizing (or optimal) auctions. It is well-known from Myerson’s (1981) seminal paper that the optimal design of auctions depends critically on the distribution of latent valuations. On the contrary, the mechanism design approach has little practical value if the primitives can not be recovered from typical auction data. The first paper which estimates the optimal mechanism in an auction environment is Paarsch (1997) which found that the reserve price used in British Columbian timber sales was far below the optimal one. For a single object for sale in the symmetric IPV setup, the optimal design problem reduces to the choice of the reserve price in a second-price auction. In multi-object setups, the computation of optimal auctions is an open issue when buyers’ valuations are multidimensional. We know from Hart and Reny (2012) and Hart and Nisan (2012) that the structure of optimal mechanisms is not understood, even, e.g., in one of the simplest possible multi-object setting: two goods and a single buyer who has independently distributed values for the goods and additive valuations. In the same vein as Hart and Nisan (2012), we thus limit ourselves to ‘simple’ auction design issues: that of selling the lots separately or as a single bundle, and under which reserve price. We strongly emphasize that those counterfactual exercises were not considered in B&BP because of the lack of knowledge (when those papers were written) on equilibrium behavior in sequential auctions with multi-unit demands. Once the distribution of latent valuations is known, then the results from Lamy (2012) provide equilibrium predictions for two-stage sequential auctions with any possible reserve price.

Similarly to Brendstrup (2007), we develop a nonparametric method to estimate the distribution of latent valuations. More precisely, our estimation part is confined to ID-sampling schemes, which stands in contrast with our identification part which covers also much more general multi-unit demand schemes. As in Brendstrup (2007), we could limit ourselves to the winning price at the last stage and exploit the aforementioned one-to-one mapping between the empirical distribution of the last stage winning bids and the distribution that characterizes how buyers’ valuations are drawn. However, we do not solely revisit Brendstrup (2007), but we also propose a nonparametric estimation procedure that also uses the winning price at the first stage, an estimation strategy that is strongly supported by Monte Carlo exper-

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5 The very limited size of the data set we use as an application prevents the estimation of a (truly) bivariate density.
ments. We then implement this approach to a data set from a tobacco warehouse located in the southeastern United States and which has been previously analyzed by Austin and Katzman (2002). Our main insight is that the chosen reserve prices seem broadly consistent with revenue maximization from the sellers’ side. We also establish that bundling would not be profitable and even in the counterfactuals with a smaller number of potential buyers than the one that prevails in those auctions. To the best of our knowledge, we are the first to investigate in an auction framework whether bundling is profitable or not with a fully nonparametric approach.\footnote{Brendstrup and Paarsch (2007) investigate whether bundling is profitable with a semiparametric approach for two-stage sequential auctions of fully heterogenous objects and with additive valuations.}

This paper is organized as follows. We develop the model in Section 2 and present the main results from Lamy (2012). We show in Section 3 that B&BP’s identification strategy is not correct and that non-identification may occur even if equilibrium behavior is assumed in the first stage. We then show how identification is restored under mild conditions that guarantee equilibrium uniqueness. In Section 4, we characterize the optimal reserve price for ‘simple’ auctions. Section 5 is devoted to estimation while Section 6 summarizes results from some Monte Carlo experiments. In Section 7, we present an application of our methodology to tobacco auctions. We conclude in Section 8. An Appendix gathers some technical proofs.

## 2 The model

We consider Lamy’s (2012) model of two-stage sequential second-price auction with multi-unit demands under the symmetric independent private-values paradigm. We make thus the following assumptions:

**A1.** The auction consists of two stages, at each stage of which a given amount of an homogenous good is sold through a second-price auction with a common reserve price $r \geq 0$ per amount of good. Without loss of generality we normalize the amount auctioned at the first stage as 1 while $s$ denotes the amount of good at the second stage. Next the variable $s$ is referred to as the weight ratio and we assume that $s \in [0, 1]$.

**A2.** There are $n \geq 2$ (potential) risk-neutral buyers.

**A3.** Each buyer’s (privately known) type $x = (x_1, x_2) \in \mathbb{R}^2_+$ with $x_1 \geq x_2$ is drawn...
independently from the publicly known bivariate distribution $F^*(..)$ on its support $\mathcal{T}$ which is a subset of $\{(x_1, x_2) \in [0, \bar{x}]^2 : x_1 \geq x_2\}$. If the buyer wins at both stages [resp. only at the first stage, only at the second stage], then his valuation for the lot(s) he obtains is equal to $x_1 + s \cdot x_2$ [resp. $x_1, s \cdot x_1$]. Next the high valuation $x_1$ [resp. the low valuation $x_2$] is referred to as the first [resp. second] valuation. The distribution of $x_1$, denoted by $F_1(\cdot)$, is assumed to be atomless, i.e. $\forall x \rightarrow F_1(x) = F^*(x, x)$ is continuous, and with a continuous positive density $f_1$ on its support $[\underline{x}, \bar{x}]$.

Let $F_2(\cdot|x_1)$ denote the CDF of the second valuation $x_2$ conditional on the realization of the first valuation $x_1$.\footnote{See Lamy (2012) for a proper definition of those conditional distributions.} We have then $F^*(x_1, x_2) = \int_{0}^{x_1} F_2(x_2|u) dF_1(u)$. For any $x_2$, then we have $\frac{\partial F^*(x_1,x_2)}{\partial x_1} = F_2(x_2|x_1) \cdot f_1(x_1)$ for almost every $x_1 \geq x_2$.

Next we always use lowercase letters for corresponding densities if they are well-defined.

A4.\footnote{A4 is a simple condition that is useful to present some of the results in Lamy (2012) in a simple way. However, it is far from being necessary for the results we present below and thus can be relaxed.} For any $x, x'$ with $\underline{x} \leq x' < x \leq \bar{x}$, the CDF $F_2(\cdot|x)$ dominates $F_2(\cdot|x')$ according to first order stochastic dominance, i.e. $F_2(y|x) \leq F_2(y|x')$ for any $y$.

At various places, we impose more structure on the sampling schemes: as in Katzman (1999) and B&BP, we then assume that the different valuations for a given buyer come from independent draws from the same underlying CDF. We then add to A3 the supplementary assumption A3b:

A3b. The valuations of any given potential buyer results from two independent draws from an atomless cumulative distribution function $F$ with a continuous positive density $f$ on its support $[\underline{x}, \bar{x}]$: the first [resp. second] valuation corresponds to the highest [resp. lowest] draw.

This additional structure is called next the ID-sampling schemes. Under A3b, we have the underlying restriction $F_2(x_2|x_1) = [F_1(x_2)/F_1(x_1)]^{1/2}$ if $x_1 > \underline{x}$. On the contrary, without this restriction, we allow general forms for $F_2(\cdot|\cdot)$ given $F_1(\cdot)$.

Note that the condition A4 always holds under ID-sampling schemes.

A5. The draws of potential buyers are mutually independent.

A6. The transaction (or winning) price in the last stage equals to the second-highest valuation for the remaining lot.
A7. A sequence of identical auctions is observed. At both stages we observe whether the lot is sold or not and the transaction price is observed when the lot is sold.

Contrary to B&BP, the following equilibrium assumption is imposed for almost all our results.

A8. Valuations are private information and buyers are playing according to a symmetric pure-strategy Bayes-Nash equilibrium.

Comments: 1) $s = 1$ corresponds to the fully homogenous case which is the paradigm in both Katzman (1999) and B&BP. Allowing the weight ratio to be smaller than one is a useful generalization because many sequential auctions concern agricultural goods where lots differ in weight/quantity. Note however that we do not allow for $s > 1$, a case where nothing is known about the equilibrium set. 2) We do not allow for synergies as in Brendstrup (2006): the restriction $x_1 \geq x_2$ assumes implicitly that buyers have substitutes preferences. 3) There are various ways to model English auctions as discussed by Bikhchandani and Riley (1991), in particular depending on whether exits are publicly observed or not. For the last stage where it is a weakly dominant strategy for each buyer to bid up to his valuation, the various modeling choices do not matter under private values and were thus not discussed in B&BP. Throughout our analysis we have implicitly in mind that English auctions are modeled as second-price auctions. The discussion on how to adapt our methodology (or make it robust) to alternative modeling choices is relegated to the conclusion. 4) B&BP assume that the identity of the winner is observed at each stage. On the contrary, we emphasize that we do not need to observe winners’ identities here since we limit our analysis to two-stage sequential auctions, while buyers are ex ante symmetric and since we consider only ‘symmetric bidding heuristics’. This is not innocuous since the identities of the bidders are not always available in public data set. 5) There is a departure from the rules of two-stage sequential auctions given in A1 that is often observed in auction houses as Christie’s and Sotheby’s: when the so-called “buyer’s option” prevails, then the winner at the first stage has the option to purchase the second lot at the first auction’s price. As the equilibrium structure is not understood with the buyer’s option under our model, then there would be

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9 If there is no binding reserve price, Brendstrup (2007) does not need this information either for two-stage auctions since he also assumes that buyers are symmetric. With a binding reserve price, Brendstrup’s (2007) methodology requires to observe whether the first lot has been sold or not in the first stage.

10 This rule plays in favor of declining prices sequence as argued by Black and de Meza (1992).
little hope for developing a structural approach as we do.

Concerning identification of the model when the reserve price is binding, i.e. when \( F^*(r,r) > 0 \) [resp. \( F(r) > 0 \) under ID-sampling schemes], we naturally mean the identification of the CDF \( F^*(.,.) \) [resp. \( F(.) \)] on the domain \([r,\infty)\times[r,\infty)\) [resp. \([r,\infty)\)]. It is well-known that we cannot identify nonparametrically the distribution of latent valuations below the reserve price.

B&BP make no assumptions on the bidding behavior in all but the last stage of the auction. The unique assumption they make on the way bidders are playing the sequential auction game is A6, i.e. that the winning price at the last stage corresponds exactly to the second highest of the valuations for this final unit, an assumption which does not rely on whether bidders are risk-neutral or risk-averse, nor on the information structure about bidders’ information and beliefs (e.g., it covers both complete and incomplete information environments). However, as it will be argued in Section 3, the econometrician cannot circumvent the issue of modeling the bidding behavior in the first stage of the auction. The incomplete information (A8) and the risk-neutrality (A2) assumptions play a key role to characterize equilibrium behavior and are consequently crucial to develop a structural approach.

Below we introduce three kinds of “bidding heuristics” at the first stage.

**Bidding heuristic \( R_\rho \):** Buyers are bidding ‘randomly’: their participation and bidding decisions in the first stage do not depend on their valuations. Each buyer refrains from participating with probability \( \rho \in [0,1] \).

**Bidding heuristic M1:** Buyers are submitting active bids (or are participating) if and only if their first valuation is above \( r \). When they participate, buyers use a common bidding function that is based solely on their first valuation and that is strictly increasing.

**Bidding heuristic M2:** Buyers are submitting active bids (or are participating) if and only if their first valuation is above \( r \). When they participate, buyers use a common bidding function that is based solely on their second valuation and that is strictly increasing when their second valuation is above \( r \).

Under our bidding heuristics, note that we do not enter into the details of the bidding function. However, we emphasize that we assume that bidders are using the same bidding function under M1 and M2. The bidding heuristics M1 and M2 have an
equilibrium foundation as shown by Katzman (1999) for ID-sampling schemes and when $s = 1$ and $r = 0$. This foundation extends to general sampling schemes and to binding reserve price as shown by Lamy (2012). On the contrary, potential buyers playing according to heuristic $R_\rho$ is incompatible with any equilibrium behavior. Nevertheless, this benchmark is useful since B&BP’s methodology remains valid if buyers were following such a bidding heuristic.

With regards to our structural econometrics perspective, the main contribution of Lamy (2012) is to establish equilibrium uniqueness and that the equilibrium follows heuristic M1.

**Proposition 2.1 (Lamy (2012))** Assume A1-A8. If either $n \geq 3$ or $s < 1$, then there is a unique (symmetric pure-strategy) equilibrium $\beta : T \to R_+$ which is given by\(^{11}\)

$$
\beta(x_1, x_2) = x_1 - s \cdot \int_r^{x_1} \left[ \frac{F_1(u)}{F_1(x_1)} \right]^{n-2} F_2(u|x_1) du \equiv \psi(x_1) \quad (1)
$$

if $x_1 \geq r$ and $\beta(x_1, x_2) < r$ otherwise. The function $\psi$ is strictly increasing on $[r, x]$ and the equilibrium follows thus heuristic M1.

If $n = 2$ and $s = 1$, then there is a continuum of equilibria among which an equilibrium following heuristic M1 where $\beta(x_1, x_2) = x_1 - \int_r^{x_1} F_2(u|x_1) du$ if $x_1 \geq r$ and $\beta(x_1, x_2) < r$ otherwise, and an equilibrium following heuristic M2 where $\beta(x_1, x_2) = x_2 \lor r$ if $x_1 \geq r$ and $\beta(x_1, x_2) < r$ otherwise.

Under A3b, the expression of the equilibrium bid function (1) simplifies to

$$
\beta(x_1, x_2) = x_1 - s \cdot \int_r^{x_1} \frac{F^{2n-3}(u)}{F^{2n-3}(x_1)} du = (1 - s) \cdot x_1 + s \cdot \int_r^{x_1} ud[F^{2n-3}(u)] \quad (2)
$$

if $x_1 \geq r$.

**Remark 2.1** With respect to B&BP’s concern about imposing less structure on the auction model, we note that the equilibrium always leads to an ex post efficient assignment and thus that there is no room for mutually profitable resale activities. Consequently, our methodology does not depend on the way we could model resale.

\(^{11}\)More formally, Lamy (2012) introduces some additional mild technical assumptions in order to guarantee the regularity of the underlying conditional expectations that are used to derive the first-order conditions that are used to characterize the set of equilibria.
From Proposition 2.1, we obtain some simple restrictions that can be easily tested before running the structural estimation exercise.

**Corollary 2.2** Assume A1-A8. If either $n \geq 3$ or $s < 1$\textsuperscript{12} then

- If the first lot remains unsold, then the second lot should remain unsold.
- If the first lot is sold at the reserve price, then the second lot should either remain unsold or be sold at the reserve price.
- If the first lot is sold for a price strictly greater than the reserve price, then the second lot should be sold (possibly at the reserve price).
- When the weight ratio gets smaller, then buyers are bidding more aggressively in the first stage.

When the first lot is sold (strictly) above the reserve price, then the transaction price at the second stage can be either smaller or larger than the transaction price at the first stage. When buyers have unit-demand (i.e. $F^*(x_1, x_2) = F_1(x_1)$) and $s = 1$, then (1) simplifies to $\beta(x_1, x_2) = \int_{r}^{x_1} (u \lor r) d(F_1(u)/F_1(x_2))^{n-2}$ and prices follow thus a martingale\textsuperscript{13} when the first item is sold (strictly) above the reserve price, then the expectation of the transaction price at the second stage equals to the transaction price at the first stage. Multi-unit demands play in favor of increasing prices as already shown by Katzman (1999) and extended in Lamy (2012) to general sampling schemes and to environments with $s \leq 1$ (lots of decreasing size is an additional force playing in favor of increasing price sequences).

**Proposition 2.3 (Lamy (2012))** Under assumptions A1-A8, the sequences of prices is a submartingale in the equilibrium following heuristic M1.\textsuperscript{14}

We emphasize that all the properties gathered in Corollary 2.2 and Proposition 2.3 hinge solely on the transaction prices and can thus be easily tested from what we have assumed in A7 about the observables. Nevertheless, note that from Lamy (2012), we could derive finer predictions depending on the identities of the winners at both stages and then test them on the data as Austin and Katzman (2002) do.

\textsuperscript{12}From Lamy (2012), the first three restrictions hold for any equilibrium when $n = 2$ and $s = 1$.
\textsuperscript{13}The seminal contribution on price trends in sequential auctions is Milgrom and Weber (2000).
\textsuperscript{14}For this result, we adopt the convention that the price of a lot is $r$ if its remains unsold.
3 Identification

In this section, we investigate whether $F^*$ [resp. $F$ under ID-sampling schemes] can be identified from the CDF of the winning prices. More precisely, we let $G(P_1, P_2)(., .)$ denote the CDF of the winning prices at both stages where the variables $P_i$ correspond to the winning price at the $i$th stage, which has been assumed to be known under A7. We adopt the convention that the winning price is strictly below $r$ if the corresponding unit remains unsold. For $i = 1, 2$, we let $G_{P_i}$ [resp. $g_{P_i}$] denote the CDF [resp. PDF] of the variable $P_i$. Let $G_{P_2|P_1}(., .)$ denote the marginal distribution of $P_2$ conditional on the realization of $P_1$ which is known from $G(P_1, P_2)(., .)$. For almost every $p_1, p_2$, we have in particular

$$\frac{\partial G(P_1, P_2)(p_1, p_2)}{\partial p_1} = G_{P_2|P_1}(p_2|p_1) \cdot g_{P_1}(p_1).$$

3.1 ID-sampling schemes

In this subsection, we limit ourselves to ID-sampling schemes and we investigate whether $F$ can be identified solely from $G_{P_2}$. We focus on the simple bidding heuristics introduced in last section. Let $G^*_{P_2}$ [resp. $G^n_{P_2}$] denote the CDF of $P_2$ conditional on the fact that the first lot is sold [resp. unsold]. Let $\chi$ the probability the first lot remains unsold. Under heuristic $R_\rho$ [resp. M1 and M2], we have $\chi = \rho^n$ [resp. $\chi = [F(r)]^n$].

Consider first heuristic $R_\rho (\rho \in [0, 1])$ where the winning or losing status in the first stage does not convey any information on the valuations of the bidders. Then the CDF of the valuation for the second lot for the winning bidder (if any) corresponds to the lowest draw from a sample of two independently and identically drawn from the CDF $F$ and is thus given by $F_{w,2}(x) = 2F(x) - F^2(x)$. For a losing bidder, the valuation for the second lot corresponds to the highest draw from a sample of 2 independently and identically drawn from the CDF $F$ and is thus given by $F_{l,2}(x) = F^2(x)$. Those are special cases of the more general bijection formula between the distribution of the $l$th largest order-statistic from a sample of $m \geq l$ independently and identically distributed draws and the distribution $F(x)$ of the underlying draws, which has the form

$$F_i^m(x) = \frac{m!}{(m-l)!(l-1)!} \int_0^{F(x)} v^{m-l}(1 - v)^{l-1} dv \equiv \phi_{i,m}(F(x)).$$

This formula is the first technical step in B&BP’s analysis that allows to trace
back bidders’ valuation distributions from their bidding CDFs at the last stage conditional on a given number of units won in the earliest stages for any number of stages. Furthermore, under heuristic \( R_{\rho} \), assumption A5 guarantees that bidders’ valuations for the second lot are drawn independently. When the first lot is sold, the winning price at the last stage corresponds then to the second order statistic among \( n \) independently distributed CDFs, one being distributed according to \( F_{w,2} \) while the \( n-1 \) remaining ones according to \( F_{l,2} \). From Balakrishnan and Rao (1998), the CDF \( G_{P_2}^s \) is thus given by

\[
G_{P_2}^s(x) = \frac{1}{(n-2)!} \int_x^\infty \text{Perm} \left| \begin{array}{ccc}
F_{l,2}(v) & \ldots & F_{l,2}(v) \\
F_{l,2}(v) & \ldots & F_{l,2}(v) \\
f_{l,2}(v) & \ldots & f_{l,2}(v) \\
(1-F_{l,2}(v)) & \ldots & (1-F_{l,2}(v)) \end{array} \right| \frac{dF_{w,2}(v)}{F_{w,2}(v)} dv
\]

for any \( x \geq r \), where \( \text{Perm} \) denotes the Permanent operator that is applied here to a \( n \times n \) matrix.\(^{15}\) This is the second crucial technical step in B&BP’s analysis that links the observed winning price distribution and bidders’ valuation distributions for the second lot. In our two-stage sequential auction framework where losing bidders are symmetric, expression (4) simplifies to:

\[
G_{P_2}^s(x) = \int_x^\infty (n-1)\left[ F_{l,2}(v) \right]^{n-3} \left\{ (n-2)F_{w,2}(v)f_{l,2}(v)[1-F_{l,2}(v)] + f_{w,2}(v)[F_{l,2}(v)[1-F_{l,2}(v)] + [1-F_{w,2}(v)][f_{l,2}(v)]f_{l,2}(v) \right \} dv.
\]

After some calculation, we obtain that \( G_{P_2}^s(x) = \Psi_{R0}[F(x)] \) for any \( x \geq r \), where \( \Psi_{R0} \) is the polynomial:

\[
\Psi_{R0}[X] = 2(n-1)X^{(2n-3)} - (n-2)X^{(2n-2)} - 2(n-1)X^{(2n-1)} + (n-1)X^{2n}.
\]

With a similar calculation, we obtain that \( G_{P_2}^a(x) = \Psi_{R1}[F(x)] \) for any \( x \geq r \), where \( \Psi_{R1} \) is the polynomial \( \Psi_{R1}[X] = nX^{(2n-2)} - (n-1)X^{2n} \). Finally, we have \( G_{P_2}(x) = \rho^n \cdot \Psi_{R1}[F(x)] + (1-\rho^n) \cdot \Psi_{R0}[F(x)] \).

\(^{15}\)For a \( n \times n \) matrix \( A = (a_{ij})_{1 \leq i,j \leq n} \), the Permanent of \( A \) is given by \( \text{Perm}A = \sum_{\sigma \in \Sigma_n} \prod_{i=1}^n a_{\sigma(i),i} \), where \( \Sigma_n \) is the set of permutation of \( \{1, \ldots, n\} \).
On the contrary, under heuristic M1, the winning or losing status in the first stage does convey information with respect to the valuations of the bidders such that those technical steps that are relying on the independence of bidders’ valuations draws can not be directly applied as in B&BP. Consider heuristic M1 and now work conditional on the highest first valuation among all bidders, a variable whose realization is denoted by \( u \). Conditional on \( u \), the CDF of the valuation for the second lot is given by

\[
F_{w,2}(x|u) = \min\left\{ \frac{F(x)}{F(u)}, 1 \right\}
\]

for the winning bidder that has won the first lot in the first stage and

\[
F_{l,2}(x|u) = \min\left\{ \frac{F^2(x)}{F^2(u)}, 1 \right\}
\]

for losing bidders that have not obtained the first lot. Conditional on \( u \), the marginal valuations for the second lot are distributed independently and we can thus apply (4) to derive the (conditional) distribution of \( P_2 \) which is denoted by \( \hat{G}_{P_2}(x|u) \):

\[
\hat{G}_{P_2}(x|u) = \begin{cases} 
(n-1) \frac{F^{2n-3}(x)}{F^{2n-3}(u)} + \frac{F^{2n-2}(x)}{F^{2n-2}(u)} - (n-1) \frac{F^{2n-1}(x)}{F^{2n-1}(u)} & \text{if } x \leq u \\
1 & \text{if } x > u 
\end{cases} \tag{5}
\]

After integrating with respect to the variable \( u \) which is distributed according to \( F^{2n} \), i.e. using the fact that

\[
\Psi_{M1}[x] = \frac{2n(n-1)}{3} X^{(2n-3)} + nX^{(2n-2)} - 2n(n-1)X^{(2n-1)} + \frac{(n-1)(4n-3)}{3} X^{2n}.
\]

**Remark** Under heuristic M1, the distributions \( F_{w,2}(.) \) and \( F_{l,2}(.) \) do not correspond to \( \phi_{2,2}(F(.)) \) and \( \phi_{1,2}(F(.)) \) their counterparts under heuristic R, contrary to what B&BP have claimed. The integration of \( F_{w,2}(x|u) \) and \( F_{l,2}(x|u) \) with respect to \( u \) leads to

\[
F_{w,2}(x) = \left[ 2F(x) - F^2(x) \right]_{\text{B&B}^{\text{P}} \text{'s term: } \phi_{2,2}(F(x))} + \frac{F(x)(1 - F(x))}{2n-1} \left[ F(x) \sum_{i=1}^{2n-2} F^{i-1}(x) - (2n - 2) \right]_{\leq 0, \text{ negative bias}}
\]

\[16\text{Those insights are also valid for heuristic M2 whose analysis here will be mainly limited to the case } n = 2 \text{ where it has an equilibrium foundation.}\]
and \( F_{i,2}(x) = \frac{F^2(x)}{\phi_{1,2}(F(x))} + \frac{2}{2n-2} [F^2(x) - F^{2n}(x)] \geq 0, \) positive bias.

The above exact formulas confirm the basic intuition that a bidder who wins [loses] the first lot of the auction sequence is more likely to have a high second valuation [a low first valuation] compared to the corresponding ex ante distributions that have been considered in B&BP. Note also that we can not plug the expression of \( F_{w,2}(x) \) and \( F_{l,2}(x) \) into the expression (4) since the valuations for the second lot are correlated: it is only conditional on \( u \) that they are independent. In a nutshell, the fundamental flaw in B&BP’s methodology relies on the fact that buyers’ valuations for the last unit have no reasons to be drawn independently (except if we assume that buyers bid according to heuristic \( R_\rho \)) in the last stage, which prevents to apply the identification results from the literature on competing risks (see Athey and Haile 2002) to recover the distribution of each buyers’ marginal valuation for the last unit in English auctions once the transaction price and the identity of the winner is observed for the last stage.

Proposition 3.1 Under heuristic \( i \in \{R_\rho, M1\} \), we have \( G_{P_2}(x) = \Psi_i[F(x)] \) for any \( x \geq r \), where \( \Psi_i \) is a known and strictly increasing polynomial function from \([0, 1]\) to \([0, 1]\) and such that \( \Psi_i^{-1} \) is differentiable on \((0, 1)\).

\[
\Psi_{R_\rho}(x) = 2(n-1)X^{(2n-3)} - [(n-2)(1-2\rho^n) - 2\rho^n]X^{(2n-2)} - 2(n-1)X^{(2n-1)} + (n-1)(1-2\rho^n)X^{2n} \\
\Psi_{M1}(x) = \frac{2n(n-1)}{3}X^{(2n-3)} + nX^{(2n-2)} - 2n(n-1)X^{(2n-1)} + \frac{(n-1)(4n-3)}{3}X^{2n}
\]

Moreover, \( \Psi_{R_\rho}(x) > \Psi_{M1}(x) > \Psi_{R_1}(x) \) on \((0, 1)\) for \( n = 2 \) while \( \Psi_{R_\rho}(x) < \Psi_{M1}(x) \) on \((0, 1)\) for \( n \geq 3 \) and any \( \rho \in [0, 1] \).

If the econometrician is prepared to assume that potential buyers are bidding according to one of the heuristic \( i \in \{R_\rho, M1\} \), then, exactly as in B&BP, Proposition 3.1 guarantees that the distribution of winning bids at the second stage enables identification of the distribution of valuations through the mapping: \( F(x) = \Psi_i^{-1}[G_{P_2}(x)] \) and a nonparametric procedure as in Brendstrup (2007) can be developed. Nevertheless, another corollary of Proposition 3.1 is a non-identification result: without any assumption on the bidding behavior on the first stage, the distribution \( F(.) \) is not identified from the distribution of the winning price of the last stage. This comes
from the fact that any distribution $G_{P_2}$ for the last stage’s transaction price can be viewed as resulting either from the CDF $F_{M1}(x) = \Psi^{-1}_{M1}[G_{P_2}(x)]$ or from the CDF $F_{R,\rho}(x) = \Psi^{-1}_{R,\rho}[G_{P_2}(x)]$ (where $\rho$ can take any value in $[0,1]$) where the CDFs $F_i$, $i \in \{R, M1\}$, are distinct.

**Corollary 3.2 (General non-identification)** Under assumptions A1-A6 and under ID-sampling schemes, $F : [r, \infty) \rightarrow [0, 1]$ is not identified from the distribution of the winning price of the last stage.

From Proposition 2.1, there exists an equilibrium that is consistent with heuristic $M2$ when $n = 2$ and $s = 1$. Similarly to what we have done under heuristic $M1$, we now show that any CDF for the winning price at the last stage can be viewed as resulting from an equilibrium under heuristic $M2$. Consider heuristic $M2$ and now work conditional on the highest second valuation among all bidders, a variable which is denoted by $t$. Conditional on $t > r$, the second valuation of the winning bidder, i.e. the bidder that has won the first lot in the first stage, is $t$ under heuristic $M2$, then the CDF of the valuation for the second lot is given by $F_{w,2}(x|t) = 1$ if $x \geq t$ for this winning bidder and $F_{l,2}(x|t) = (2F(x)F(t) - F^2(t))/(2F(t) - F^2(t))$ if $x < t$ for losing bidders that have not obtained the first lot. Conditional on $t > r$, the $n$ valuations for the second lot are distributed independently and we can thus apply (4) to derive the distribution of $P_2$ which is denoted by $\tilde{G}_{P_2}(.,|t)$. For $n = 2$, it leads to:

$$\tilde{G}_{P_2}(x|t) = F_{w,2}(x|t) + F_{l,2}(x|t) - F_{w,2}(x|t) \cdot F_{l,2}(x|t) = \begin{cases} \frac{F^2(x)}{2F(t) - F^2(t)} & \text{if } x < t \\ 1 & \text{if } x \geq t \end{cases}. \quad (6)$$

Remark that the CDF $\tilde{G}_{P_2}(.,|t)$ has an atom at $t$. Conditional on $t \leq r$ and when $n = 2$, then the second lot remains unsold or is sold at the reserve price and we have thus $\tilde{G}_{P_2}(x|t) = 1$ for any $x \geq r$. From the integration with respect to the variable $t$ which is distributed according to $(2F - F^2)^2$, i.e. using the fact that $G_{P_2}(x) = \int \tilde{G}_{P_2}(x|t)d[(2F(t) - F^2(t))^2]$, we obtain (when $n = 2$) that $G_{P_2}(x) = \Psi_{M2}[F(x)]$ for any $x \geq r$, where $\Psi_{M2}$ is the polynomial

$$\Psi_{M2}[X] = 6X^2 - 8X^3 + 3X^4.$$
Then the same logic that leads to corollary 3.2 leads to the following non-identification result.

**Corollary 3.3 (Non-identification under equilibrium behavior)** Under assumptions A1-A6 and A8, under ID-sampling schemes and when \( n = 2 \) and \( s = 1 \), \( F : [r, \infty) \rightarrow [0,1] \) is not identified from the distribution of the winning price of the last stage.

Corollary 3.3 is a much stronger than Corollary 3.2 insofar as equilibrium behavior (A8) is assumed in the former non-identification result. Nevertheless, Corollary 3.3 is limited to \( n = 2 \) and \( s = 1 \).

**Remark 3.1** Our non-identification results extent if we consider identification from the winning price of the last stage and the information whether the first lot has been sold or not, i.e. \( F \) can not be identified from \( G_{P_2}^n, G_{\tilde{P}_2}^n \) and \( \chi \). This comes from the fact that \( G_{P_2}^n \) and \( \chi \) are always the same under heuristic M1 and M2 and such do not bring any additional information in terms of identification: \( G_{P_2}^n(x) = 1 \) if \( x \geq r \) and \( \chi = [F(r)]^n \).

**Remark 3.2** The analysis in B&BP covers mainly the case without reserve and is valid if potential buyers are following heuristic R0. In the extension to a binding reserve price \( r \), Brendstrup’s (2007) analysis assumes implicitly that buyers are following heuristic R_{F(r)}.

We can then revisit Example 1 in Brendstrup (2007): the two bidders and two homogenous units case with no reserve price. \( F(\cdot) \) is uniquely characterized as an implicit function by the equation \( G_{P_2}(x) = \Psi_i[F(x)] \) for the different heuristics \( i \in \{ R_0, M1, M2 \} \). In Figure 1 the functions \( \Psi_i^{-1}[X], i \in \{ R0, M1, M2 \} \) are depicted, equivalently it gives the expression of \( F(x) \) as a function of \( G_{P_2}(x) \) for our different bidding heuristics. The differences between two curves \( i \) and \( j \) corresponds then to the bias when one assumes a wrong heuristic \( i \) while the true bidding heuristic is \( j \).

The graphs show that the bias is especially important between \( \Psi_{R0}^{-1} \) and \( \Psi_{M2}^{-1} \). If one assumes heuristic R0, as it is implicitly the case in B&BP, while the true bidding

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17Another source of non-identification would emerge if we do not assume an ‘incomplete information’ structure (as under A8) but allow also bidding under complete information. Then a similar non-identification result as corollary 3.3 could be derived for any number of bidders while still restricting attention to bidding behaviors that are Nash equilibria with risk neutral bidders.
heuristic is either M1 or M2, then the probability $F(x)$ is underestimated for any $x \in (\underline{x}, \overline{x})$. The range of the bias is considerable since it is greater than 0.1 for more than one third of the support in the case of heuristic M2. Note that the sign of the misspecification bias if one assumes heuristic $R_0$ while the true bidding heuristic is M1 changes when $n \geq 3$ as established by Proposition 3.1: it means that B&BP’s methodology overestimates the probability $F(x)$ for any $x \in (\underline{x}, \overline{x})$ or equivalently underestimate the distribution $F$ according to first-order stochastic dominance.

When $n \geq 3$ or $s < 1$, then the equilibrium is unique and follows heuristic M1 which guarantees identification.

**Corollary 3.4 (Identification under equilibrium behavior and mild restrictions)**

*Under assumptions A1-A6, A8 and under ID-sampling schemes, if either $n \geq 3$ or $s < 1$, then $F : [r, \infty) \rightarrow [0,1]$ is identified from the distribution of the winning price of the last stage.*

From our analysis above, it is clear that the crucial point is not that the equilibrium corresponds exactly to the one derived in Proposition 2.1 but rather that it follows heuristic M1. In particular, from Lamy (2012), we could adapt straightforwardly our methodology under ID-sampling schemes to environments with nondecreasing absolute risk aversion since there is an equilibrium following heuristic M1.
Note that the first stage’s bids could then be used to identify risk-aversion in the same vein as in Lu and Perrigne (2008).\textsuperscript{18}

### 3.2 General sampling schemes

Under A1-A6, A8 and under general sampling schemes, $F^*(\cdots)$ can not be identified from the distribution of the winning price of the last stage: any winning price distribution generated from a CDF $F^*(\cdots)$ can be alternatively viewed as coming from an ID-sampling scheme from the marginal distribution $\Psi_{M1}[G_{P1}(\cdot)]$. Contrary to B&BP, we thus consider next identification from the joint distribution of the bids at both stages.

**Proposition 3.5** Under A1-A8 and when either $n \geq 3$ or $s < 1$, the distribution $F^*: [r, \infty) \times [r, \infty) \to [0, 1]$ is identified from the winning price at both stages.

**Proof** After having noted that $G_{P1}(\cdot)$ is atomless on $[r, \infty)$ under heuristic M1, then similar derivations as in the previous subsection leads to: for any $b_1 \geq r$

\[ G_{P1}(b_1) = \phi_{2,n}(F1(\beta^{-1}(b_1))) \quad \text{and} \]

\[ G_{P2|P1}(b_2|b_1) = \begin{cases} \frac{F1(b_2)}{F1(\beta^{-1}(b_1))} \cdot \frac{[\int_{\beta^{-1}(b_1)}^{\beta^{-1}(b_1)} F2(b_2|s) \cdot d[F1(s)]]}{1-F1(\beta^{-1}(b_1))} & \text{if } b_2 < \beta^{-1}(b_1) \\ 1 & \text{if } b_2 \geq \beta^{-1}(b_1) \end{cases} \]

for respectively the first and second stages and where $\beta$ is given by Eq. (1). If $\beta$ were known, then $F1(.)$ would be identified from Eq. (7) by

\[ F1(x) = \phi^{-1}_{2,n}(F_{P1}(\beta(x))). \]

Subsequently, we could also identify $\int_{\beta^{-1}(b_1)}^{x} F2(b_2|s) \cdot d[F1(s)]$ from Eq. (8) for any $b_1, b_2$. The derivation with respect to $\beta^{-1}(b_1)$ would lead to the identification

\textsuperscript{18}When buyers’ utility function $U(.)$ is a CARA utility function, i.e. $U(x) = 1-e^{-\alpha x}$ with $\alpha > 0$, then the equilibrium bid function $\beta$ under heuristic M1 satisfies $\frac{\partial}{\partial x_1}(x_1, x_2) = \frac{1}{\alpha} \cdot \frac{(2n-3)F(x_1)}{F(x_1)}$. We can identify $f$ and $F$ from the second stage’s bids and then identify $\beta$ from the first stage bids and finally identify $\alpha$. For general (nonparametric) utility functions, whether the utility function can be identified or not is an open issue.
of \( F_2(b_2 | \beta^{-1}(b_1))f_1(\beta^{-1}(b_1)) \) and then to \( F_2(x_1 | x_1) \) for any \( x_1 \in [x, \bar{x}] \) (note that \( f_1(x_1) > 0 \)). Since \( \frac{\partial F^*}{\partial x_1}(x_1, x_2) = F_2(x_2 | x_1)f_1(x_1) \) almost everywhere, \( F^*(., .) \) would thus be identified and we would be done.

It remains to show that \( \beta \) is actually identified. For any \( b \in [x, \beta(\bar{x})] \), \( \beta^{-1}(b) \) corresponds to the upper bound of the support of the distribution \( G_{P_2|P_1}(.|b) \).\(^{19}\) \( \beta^{-1} \) is identified and thus \( \beta \).\(^{20}\)Q.E.D.

4 Auction design

Once we have identified the model (completely if the reserve price is not binding or only partially if the reserve price is binding), then we can answer some auction design problems. Let \( X_S \in [0, \bar{x}] \) denote the valuation (per amount of good) of the seller which is assumed to be the same for the first and the second lot. Let \( \Pi(r) \) [resp. \( \Pi_{\text{band}}(r) \)] denote the expected payoff of the seller as a function of the reserve price \( r \) in two-stage sequential auctions [resp. in auctions where the two lots are bundled in a single package]. Similarly, we let \( W(r) \) [resp. \( W_{\text{band}}(r) \)] denote the corresponding expected welfare. Additionally to the reserve price, those functions depend on the distribution of the latent valuations \( F^*(., .) \) but also on \( n, s \) and \( X_S \) which are taken as exogenous.

Thanks to Proposition 2.1, the estimation of the distribution of the latent valuations allows us to run counterfactuals in sequential auctions with various reserve prices. More precisely, if we identify \( F^*(., .) \) on \([r, \infty) \times [r, \infty)\), then we can derive the equilibrium bid function from (1) and then the corresponding seller’s expected payoff \( \Pi(\bar{r}) \) [welfare \( W_{\text{band}}(\bar{r}) \)] for any reserve price \( \bar{r} \geq r \). In particular, we can identify the optimal reserve price above \( r \), i.e. \( \arg \max_{\bar{r} \geq r} \Pi(\bar{r}) \). Furthermore, if this later value equals \( r \) and if \( \lim_{\bar{r} \to r^+} \frac{d\Pi(\bar{r})}{d\bar{r}} < 0 \), then we can infer by continuity that the optimal reserve price lies strictly below \( r \).

On the contrary, we emphasize that we can not identify the seller’s expected

\(^{19}\)When the model involves “truly multi-unit demand”, in particular if \( F_2(x_2 | x_1) < 1 \) if \( x_2 < x_1 \), then \( \beta^{-1}(b) \) corresponds also to an atom of the distribution \( G_{P_2|P_1}(.|b) \). This atom corresponds to the event where the winner of the first stage also wins the auction at the second stage. We conjecture that from a practitioner’s perspective the ‘atom property’ could help considerably estimation.

\(^{20}\)If the identities of the winners were observed, then \( \beta \) could be identified in a more direct way. In the events where the winner is the same in both stages, then heuristic M1 guarantees that the highest losing bidder should be the same in both stages: we obtain then that \( P_1 = \beta(P_2) \). This observation could be of great help to enhance estimation.
payoff $\Pi_{\text{band}}(\tilde{r})$ in auctions with bundling for any reserve price $\tilde{r} \geq r$ if the reserve price $r$ is binding. In order to identify those expected payoffs, we have to recover $F_{\text{band}}$ the CDF of the valuation for the bundle, i.e. $x_1 + sx_2$, on $[\tilde{r}, \infty)$. We have $F_{\text{band}}(x) = F^*(\frac{x}{1+s}) + \int_{\frac{x}{1+s}}^{\infty} \frac{\partial F^*(x,s|u)}{\partial x_1} du$ or equivalently

$$F_{\text{band}}(x) = F1(\frac{x}{1+s}) + \int_{\frac{x}{1+s}}^{x-r} F2(\frac{x-u}{s}) d[F1(u)] + \int_{x-r}^{x} F2(\frac{x-u}{s}) d[F1(u)]$$

for any $x \geq (1 + s) \cdot r$. While the first two terms in (9) depend solely on the way the valuations are distributed above $r$, the last term depends obviously on the way the second valuation is distributed below $r$ and this remains true under the ID-sampling restriction. In the application we consider in Section 7, 18% of the auctions of our (restricted) sample have no reserve price at all which enables us to estimate the entire distribution of latent valuations and thus to answer policy design issues with respect to bundling.\footnote{If we identify $F^*$ on $[r, \infty) \times [r, \infty)$ where $r$ is a binding reserve price, then we can nevertheless establish some bounds on $F_{\text{band}}$. More precisely, we have $F1(\frac{x}{1+s}) + \int_{\frac{x}{1+s}}^{x-r} F2(\frac{x-u}{s}) d[F1(u)] \leq F_{\text{band}}(x) \leq F1(\frac{x}{1+s}) + \int_{\frac{x}{1+s}}^{x-r} F2(\frac{x-u}{s}) d[F1(u)] + \int_{x-r}^{x} F2(r|u) d[F1(u)]$ for any $x \geq (1 + s) \cdot r$. In the same way as Haile and Tamer (2003), those bounds could be used to establish some bounds on the optimal reserve price with bundling.}

A priori, we have no idea whether it is profitable for the seller to bundle or not the two lots into a single package. The main results from the theoretical literature are of limited scope: when $n = 2$, bundling is always profitable in multi-object (generalized) Vickrey auctions where the reserve price is set at the seller’s valuation (Krishna 2002) and so, as a special case, in our model when $r = X_S$. Chakraborty (1999) considers a slightly different model with two heterogeneous goods and buyers having additive preferences and establishes that separate Vickrey (or equivalently efficient) auctions become profitable as $n$ gets larger compared to the single Vickrey auction with bundling. On the contrary, if the seller is free to set any reserve price, then nothing is known about the profitability of bundling even if $n = 2$. As illustrated by the simulations reported in Table 1, running separate auctions may outperform bundling when $n = 2$ if reserve prices are set at their optimal level, which are denoted by $r^*$ and $r^*_{\text{band}}$ for sequential auctions and single auctions for the two lot package, respectively. More generally, we can not hinge on any theoretical guidelines.
concerning bundling decisions or, said differently, the profitability of bundling is an empirical question.\textsuperscript{22}

Furthermore, in terms of policy design, a welfare maximizing perspective may be the most relevant one for some issues. If we have in mind that it is the auctioneer (instead of the seller) who chooses whether bundling is allowed or not while she cares about the entry in her auction house, then the welfare criterion is probably the one that matters.\textsuperscript{23} If we limit ourselves to auctions where the reserve price is set at the seller’s valuation, then it is straightforward that bundling is always detrimental to the welfare. Nevertheless, the answer is no longer clear-cut if sellers are free to choose their own reserve price. In our simulations, it occurs in particular that the optimal reserve price is much lower under bundling than under sequential auctions, which suggests that bundling might be welfare maximizing. Nevertheless, we have always found in our simulations that the welfare is larger when the two units are sold separately.\textsuperscript{24} Without theoretical guidelines, we would have to check whether it holds in each particular application.

We next characterize the optimal reserve price in simple auction designs. Let

\begin{equation}
\xi(x) := x - \frac{1 - F_1(x)}{f_1(x)} \left[ \frac{1 + s \cdot \left( 1 - A(x) + A(x) \cdot (N - 1) \frac{(1 - F_1(x))}{F_1(x)} \right)}{1 + s \cdot \left( 1 - A(x) + A(x) \cdot (N - 1) \frac{(1 - F_1(x))}{F_1(x)} + \frac{dA(x)}{dx} \cdot \frac{(1 - F_1(x))}{f_1(x)} \right)} \right]
\end{equation}

(10)

where $A(x) := \int_{x}^{x'} F_2(x|u) \frac{dF_1(u)}{1 - F_1(x)}$ denotes the probability that the second valuation of a bidder that has a first valuation above $x$ lies below $x$, and let

\begin{equation}
\xi_{\text{band}}(x) := x - \frac{1 - F_{\text{band}}(x)}{f_{\text{band}}(x)}.
\end{equation}

Next assumption guarantees that $\Pi(.)$ and $\Pi_{\text{band}}(.)$ are strictly quasi-concave such

\textsuperscript{22}Jehiel et al. (2007) is an exception in the literature by considering bundling with a mechanism design perspective, including in particular strategic reserve prices, in a model with heterogeneous objects and buyers having additive valuations. However, their results are useless when one wants to compare separate auctions to auctions with bundling. They only show that under mild conditions, the optimal mechanism is somehow in the middle.

\textsuperscript{23}The auctioneer organizes a large set of auctions and her objective may then better coincide with welfare maximization because of endogenous entry as in Levin and Smith’s (1994). On the contrary, each individual seller takes as given the number of buyers choosing to come to the auction house.

\textsuperscript{24}With only one buyer and in Chakraborty’s (1999) model, Hart and Nisan (2012) have examples where the seller’s revenue maximizing price for the bundle raises an higher expected welfare than the mechanism where each good is sold separately at the seller’s revenue maximizing prices.
that first-order conditions characterize the optimal reserves \( r^* \) and \( r^*_{\text{band}} \).

**A9.** \( \xi(.) \) and \( \xi_{\text{band}}(.) \) are strictly increasing on \([X_S \vee x, \overline{x}]\).

Under A9, we have then \( r^* = \xi^{-1}(X_S) \) and \( r^*_{\text{band}} = \xi_{\text{band}}^{-1}(X_S) \). Under ID-sampling schemes, those formulas simplify to

\[
X_S = r^* - 1 - \frac{[F(r^*)]^2}{2F(r^*)f(r^*)}\left[\frac{1 + s \cdot (2N - 1 - \frac{1}{1+F(r^*)})}{1 + s \cdot (2N - 1)}\cdot \frac{1-F(r^*)}{F(r^*)}\right]
\]

and

\[
X_S = r^*_{\text{band}} - \frac{1 - [F(r^*_{\text{band}})]^2 - 2 \int_{r^*_{\text{band}}}^{r^*} F(r^* - u) f(u) du}{\frac{2}{s} \int_{r^*_{\text{band}}}^{r^*} f(r^* - u) f(u) du}.
\]

In the symmetric IPV model, it is well-known that the optimal reserve in single-good auctions does not depend on the number of buyers. Next we are interested in how \( r^* \) varies with respect to the exogenous variables \( n \) and also \( s \). For this we introduce an additional technical assumption.

**A10.** For any \( x \in (\underline{x}, \overline{x}) \), the CDF \( F1(.) \) is strictly greater than \( F2(\cdot|x) \) according to hazard rate dominance, i.e. \( F2(r|x) < F1(r) \) for any \( \underline{x} < r < x \).

**Remark 4.1** We can check that A10 always holds under A3b: the inequality \( \frac{F2(r|u)}{1-F2(r|u)} > \frac{1-F1(r)}{F(u)-F(r)} \) is then equivalent to \( \frac{1}{F(u)-F(r)} > \frac{2F(r)}{1-[F(r)]^2} \), which holds once \( r < u \leq \overline{x} \).

**Proposition 4.1** Under A1-A10, then the optimal reserve price \( r^* \) is decreasing in \( s \) and increasing in \( n \).

Contrary to single-good auctions, we note that the optimal reserve depends on the number of bidders. The intuition is the following. If there is only one bidder, the optimal reserve is lower than in the case where this bidder has unit-demand and with the same distribution for his first valuation because the seller is also concerned by his demand for the second lot and wish to sell the second lot: formally, this results from the fact that the term in the bracket in Eq. (10) is smaller than one. When the number of buyers gets large, then the multi-unit effect becomes relatively less

\(^{25}\)See Appendix B for details on the calculations.
\(^{26}\)We implicitly assume that \( f2(\cdot|x) \) is well-defined on \([0, x] \), and in particular that \( F2(\cdot|x) \) contains no atoms. Note that A10 excludes the unit-demand and the flat-demand cases where the optimal reserve prices depend neither on \( n \) nor on \( s \).
\(^{27}\)This results from \( \frac{dA(u)}{dx} \cdot \frac{(1-F1(x))}{F1(x)} > 0 \) (see the proof of Proposition 4.1).
important (the term in the bracket in Eq. (10) gets close to one) and we get closer to the unit-demand formula and thus to a larger reserve. The intuition is the same when the weight ratio gets smaller. In particular, when $s$ goes to zero we are back to the unit-demand case and so to a larger optimal reserve because there is no concern for attracting bids for the second lot.

Table 1: Policy design simulations under ID-sampling schemes with $F(x) = x^3$ on $[0, 1]$ and $X_S = 0$

<table>
<thead>
<tr>
<th></th>
<th>$n = 2$</th>
<th></th>
<th>$n = 3$</th>
<th></th>
<th>$n = 5$</th>
<th></th>
<th>$n = 10$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$s$</td>
<td>0.15</td>
<td>0.25</td>
<td>0.5</td>
<td>0.75</td>
<td>1†</td>
<td>0.5</td>
<td>1</td>
</tr>
<tr>
<td>$r^*$</td>
<td>0.714</td>
<td>0.707</td>
<td>0.702</td>
<td>0.700</td>
<td>0.698</td>
<td>0.710</td>
<td>0.703</td>
</tr>
<tr>
<td>$r^*_{band}$</td>
<td>0.706</td>
<td>0.682</td>
<td>0.649</td>
<td>0.622</td>
<td>0.603</td>
<td>0.649</td>
<td>0.603</td>
</tr>
<tr>
<td>$\Pi(r^*)$</td>
<td>0.862</td>
<td>0.949</td>
<td>1.094</td>
<td>1.238</td>
<td>1.383</td>
<td>1.239</td>
<td>1.603</td>
</tr>
<tr>
<td>$\Pi_{band}(r^*_{band})$</td>
<td>0.864</td>
<td>0.951</td>
<td>1.092</td>
<td>1.231</td>
<td>1.369</td>
<td>1.197</td>
<td>1.520</td>
</tr>
<tr>
<td>$W(r^*)$</td>
<td>0.986</td>
<td>1.100</td>
<td>1.291</td>
<td>1.481</td>
<td>1.672</td>
<td>1.382</td>
<td>1.818</td>
</tr>
<tr>
<td>$W_{band}(r^*_{band})$</td>
<td>0.979</td>
<td>1.085</td>
<td>1.265</td>
<td>1.447</td>
<td>1.629</td>
<td>1.325</td>
<td>1.718</td>
</tr>
<tr>
<td>$\Pi(X_S)$</td>
<td>0.837</td>
<td>0.905</td>
<td>1.019</td>
<td>1.132</td>
<td>1.246</td>
<td>1.232</td>
<td>1.590</td>
</tr>
<tr>
<td>$\Pi_{band}(X_S)$</td>
<td>0.850</td>
<td>0.935</td>
<td>1.073</td>
<td>1.210</td>
<td>1.346</td>
<td>1.195</td>
<td>1.518</td>
</tr>
<tr>
<td>$W(X_S)$</td>
<td>1.006</td>
<td>1.131</td>
<td>1.338</td>
<td>1.546</td>
<td>1.754</td>
<td>1.391</td>
<td>1.836</td>
</tr>
<tr>
<td>$W_{band}(X_S)$</td>
<td>0.993</td>
<td>1.101</td>
<td>1.283</td>
<td>1.468</td>
<td>1.654</td>
<td>1.327</td>
<td>1.722</td>
</tr>
</tbody>
</table>

†: in the case $n = 2$ and $s = 1$, we assume that the equilibrium following heuristic M1 is played.

The results from our simulations deserve some comments with regards to an empirical perspective where we may wish to test whether the reserve price is set optimally. From the third line in Table 1, we see that the range of the optimal reserves when $n$ and $s$ vary is very limited under sequential auctions. Once $n \geq 5$, then the weight ratio has only a vanishing impact on the optimal reserve. On the whole, even if sellers are choosing an optimal reserve and given that there is some noise in the way the reserve prices are set (as in our data set) due possibly to the heterogeneity in the seller’s valuations, then there is few chances to detect a significant correlation between either the number of buyers or the weight ratio and the reserve price policy.
5 Estimation

In this section, we set up the estimation method under ID-sampling schemes. We do more than simply fixing Brendstup’s (2007) procedure to account for the selecting bias that arises under heuristic M1 with respect to heuristic R0. Contrary to Brendstup (2007), we propose a nonparametric estimation procedure that uses also the first stage’s bids in order to gain in term of efficiency as it will be argued in Section 6. Let $T$ denote the total number of observations. Each observation $t \in \{1, \ldots, T\}$ consists of a pair of prices $(B^1_t, B^2_t)$ where $B^1_t$ corresponds to the winning price at the $i^{th}$ stage. In a first step, we assume (as it is implicitly the case in B&BP) that the reserve price $r$ and the weight ratio $s$ remain constant over the sample. When the lot remains unsold at stage $i$, we let $B^i_t = -1$.

For $i = 1, 2$, the functions $G_{P_i}$ and $g_{P_i}$ can be estimated respectively by its empirical distribution and by standard kernel estimation techniques:

$$
\hat{G}_{P_i}(b) = \frac{1}{T} \sum_{t=1}^{T} 1(B^i_t \leq b) \quad \text{and} \quad \hat{g}_{P_i}(b) = \frac{1}{h_i^T} \sum_{t=1}^{T} K_{i^g}(\frac{b - B^i_t}{h_i^g}),
$$

where $h_i^g > 0$ is a bandwidth and $K_{i^g}(\cdot)$ is a kernel with the support $[-1, 1]$. Our standard kernel estimator for the density $\hat{g}_{P_i}(b)$ is biased for $b$ at the lower bound of the bidding support, i.e. for $b \in [r, r + h_i^g]$. To avoid this problem, we could use a method of boundary correction (see e.g. Karunamuni et al. (2005) for a recent contribution on generalized reflection method).

Estimation from the second stage

From Proposition 3.1, bids at the second stage can be used to give a nonparametric estimate of $F : [r, \infty) \rightarrow [0, 1]$ exactly as in Brendstrup (2007) provided that we replace his function $\Psi \equiv \Psi_{R0}$ with the one which corresponds to heuristic M1, i.e. $\Psi_{M1}$. Let $\hat{F}^2(x)$ and $\hat{f}^2(x)$ denote the estimator of the CDF and PDF from this stage:

$$
\hat{F}^2(x) = \Psi^{-1}_{M1}(\hat{G}_{P_2}(x)) \quad \text{and} \quad \hat{f}^2(x) = \frac{\hat{g}_{P_2}(x)}{\Psi'_{M1}(\Psi^{-1}_{M1}(\hat{G}_{P_2}(x)))}.
$$

With appropriate smoothness assumptions (the CDF $F$ being three times differentiable on $(x, \overline{x})$), then the asymptotic statistical properties of those estimators (in particular consistency) are the same as in Brendstrup (2007) since $\Psi^{-1}_{M1}$ is differentiable on $(0, 1)$ exactly as $\Psi^{-1}_{R0}$ was in his analysis.
Remark 5.1 The estimator $\hat{F}^2(.)$ is defined as the composition of the bijection $\Psi_{M1}^{-1}(.)$ on $[0,1]$ with a function which is a CDF as it corresponds to the empirical distribution of the second stage winning bids. The function $\hat{F}^2(.)$ is thus a CDF.

Estimation from the first stage

The empirical counterpart of the expression of the equilibrium bid function (2) can be used to build a set of ‘pseudo-valuations’ in the same vein as Guerre et al.’s (2000) two-stage estimator.

The estimation from the second stage winning bids gives us a natural estimator $\hat{\psi}$ for $\psi: [r, \infty) \rightarrow [r, \infty)$: for $x \geq r$, we let $\hat{\psi}(x) = x$ if $\hat{F}^2(x) = 0$ and

$$\hat{\psi}(x) = x - s \cdot \int_r^x \frac{[\hat{F}^2(u)]^{2n-3}}{[\hat{F}^2(x)]^{2n-3}} du = (1 - s) \cdot x + s \cdot \int_r^x ud\left[\frac{\hat{F}^2(u)}{\hat{F}^2(x)}\right]^{2n-3}$$

otherwise. Since $\hat{F}^2(.)$ is a CDF as noted in Remark 5.1, the last integral in (16) is nondecreasing. If $s < 1$, we obtain then that $\hat{\psi}$ is strictly increasing and is thus a bijection from $[r, \infty)$ to $[r, \infty)$ so that $\hat{\psi}^{-1}$ is well-defined. On the contrary, if $s = 1$, then $\hat{\psi}$ is a step function where the steps occur at the bids $B_2^t$, $t = 1, \ldots, T$. This leads us to different procedures depending on whether $s < 1$ or $s = 1$.

If $s < 1$, then we define the pseudo-valuation $X^1_t$ as

$$X^1_t = \hat{\psi}^{-1}(B^1_t)$$

if $B^1_t > r$ and $X^1_t = -1$ otherwise.

The derivation with respect to the variable $x$ of the expression of the equilibrium bid function $\psi$ in (1) and the change of variable $b = \psi(x)$ leads to the equation:

$$\psi^{-1}(b) = b + \frac{1}{2n-3} \cdot \frac{D_1(b)}{d_1(b)} - \frac{1 - s}{2n-3} \cdot \frac{F(\psi^{-1}(b))}{f(\psi^{-1}(b))},$$

for any $b > r$ where $D_1$ and $d_1$ are respectively the CDF and the PDF of the bids at the first stage. Such a reparametrization of the equilibrium equation is similar to the one that first appeared in Guerre et al. (2000) for first-price auctions. The relation between the bid distribution in the first stage and the winning price distribution is given by $D_1(b) = \phi_{2,n}^{-1}(G_{P_1}(b))$ so that $D_1$ is identified from the data since $P_1$ is observed. The empirical counterpart provides the estimators:
\[ \hat{D}_1(b) = \phi_{2,n}^{-1}(\hat{G}_P(b)) \quad \text{and} \quad \hat{a}_1(b) = \frac{\hat{g}_P(b)}{\phi_{2,n}^{-1}(\hat{G}_P(b))}, \]  
(19)

**Remark 5.2** When \( s = 1 \), (18) reduces to \( \psi^{-1}(b) = b + \frac{1}{2n-3} \cdot \frac{D_1(b)}{d_1(b)} \) which allows us to express buyers’ first valuations from their bids and the elasticity of their probability of winning. Under A1-A8 and if \( s = 1 \) and \( n \geq 3 \), we obtain thus that \( F : [r, \infty) \rightarrow [0, 1] \) is identified from the winning price of the first stage under ID-sampling schemes, i.e. from the CDF \( G_P \).

Eq. (18) provides a natural nonparametric estimation path in order to use the first stage transaction prices. If \( s = 1 \), then we define the pseudo-valuation \( X_1^t \) as

\[ X_1^t = B_1^t + \frac{1}{2n-3} \cdot \frac{\hat{D}_1(B_1^t)}{\hat{d}_1(B_1^t)} \]  
(20)

if \( B_1^t > r \) and \( X_1^t = -1 \) otherwise.\(^{28}\)

We do not detail this point here but a trimming rule at the boundaries of the support is needed to avoid some bias in the same way as in Guerre et al. (2000). Then we use the pseudo sample \( \{X_1^t, t = 1, \ldots, T\} \) to estimate nonparametrically the CDF \( F_{2,n}^1 \) and PDF \( f_{2,n}^1 \) of the first valuation corresponding to the highest losing bid in the first stage for the underlying CDF \( F \):

\[ \hat{F}_{2,n}^1(x) = \frac{1}{T} \sum_{t=1}^{T} I(X_1^t \leq x) \quad \text{and} \quad \hat{f}_{2,n}^1(x) = \frac{1}{h_f T} \sum_{t=1}^{T} K_f(\frac{x - X_1^t}{h_f}), \]  
(21)

where \( h_f > 0 \) is a bandwidth and \( K_f(\cdot) \) is a kernel with the support \([-1, 1]\).

Since the first valuation for a given bidder is distributed according to the CDF \( [F(x)]^2 \), the relation between the first valuation of all bidders and the first valuation corresponding to the highest losing bidder in the first stage is \( F_{2,n}^1(x) = \phi_{2,n}([F(x)]^2) \).

Finally, the winning price from the first auction leads to the following estimator:

\(^{28}\)If \( s < 1 \), note that the empirical counterpart of (18) suggests an alternative estimation path for the pseudo-valuations. If the function \( x \rightarrow x + \frac{1}{2n-1} \cdot F(x)^2 \) is strictly increasing (as it is the case when \( s = 1 \)), then we could define \( X_1^t \) as the solution of the equation \( X_1^t + \frac{1}{2n-1} \cdot F(x_1^2) = B_1^t + \frac{1}{2n-3} \cdot \frac{\hat{D}_1(B_1^t)}{\hat{d}_1(B_1^t)}. \)
\( \hat{F}_1(x) = [\phi_{2,n}(\hat{F}_1(x))]^{\frac{1}{2}} \) and \( \hat{f}_1(x) = \frac{\hat{F}_1(x)}{2[\phi_{2,n}(\hat{F}_1(x))]^{\frac{1}{2}} \phi'_{2,n}(\phi_{2,n}(\hat{F}_1(x)))}. \) (22)

**Remark 5.3** Note that \([\phi_{2,n}(\cdot)]^{\frac{1}{2}}\) is a bijection on \([0,1]\). From the same argument as in Remark 5.1, our estimator \( \hat{F}_1(.) \) is thus also a CDF.

The statistical properties of this estimator from the first stage’s winning price, e.g. uniform consistency, can be derived exactly in the same way as in Guerre et al. (2000) due to the similarity of the estimation procedure. The unique fundamental difference with Guerre et al. (2000) comes from the fact that we do not observe the bid [pseudo valuation] distribution but only the highest losing bid [highest losing pseudo valuation] distribution which requires the uses of the transformations (19) and (22). Those transformations involve differentiable functions on \((0,1)\) such that the delta method applies.

Finally we propose to estimate the CDF and PDF of the latent valuations by combining our estimators from both stages using a weighted least squares approach.

\[
\hat{F}(x) = \text{Arg min}_u g(u)'Wg(u) \quad \text{and} \quad \hat{f}(x) = \text{Arg min}_u \gamma(u)'\Omega \gamma(u)
\]

where \(g(u)\) [resp. \(\gamma(u)\)] is a 2 dimensional vector with elements \(\hat{F}_i(u) - u\) [resp. \(\hat{f}_i(u) - u\)] and \(W\ [\Omega]\) is a weighting matrix. The weights can be chosen to minimize the variance as suggested by Brendstrup (2007).

A direct testable restriction is that both distributions \(\hat{F}_1\) and \(\hat{F}_2\) should be both consistent estimators of a common distribution. If \(\hat{F}_1\) and \(\hat{F}_2\) are not close to each other then we can suspect that the model is misspecified, e.g. because of the latent valuations do not result from an ID-sampling schemes or because buyers are risk-averse.

### 5.1 Heterogeneity across auctions

In our specific application, both the reserve price and the weight ratio vary across auctions so that the previous estimation procedure should be slightly adapted. For each observation \(t\), we let \(R_t\) the reserve price and \(S_t\) the weight ratio. Here we assume that \(S_t < 1\) as in the subsample of the data set we consider.
In order to estimate $F(x)$ for a given $x$, we restrict ourselves to the subsample of auctions with $R_t \geq x$ and run exactly the same procedure as above. For respectively $b \geq \min_t \{R_t\}$ and $b \geq \min_t \{R_t\} + h^2$, the functions $G_{P_2}$ and $g_{P_2}$ are now estimated by:

$$\hat{G}_{P_2}(b) = \frac{\sum_{t=1}^{T} \mathbf{1}(\max \{B^2_t, R_t\} \leq b) }{\sum_{t=1}^{T} \mathbf{1}(R_t \leq b) }$$

and

$$\hat{g}_{P_2}(b) = \frac{1}{h^2_g} \frac{\sum_{t=1}^{T} K^2_g \left( \frac{b - R^2_t}{h^2_g} \right) \cdot \mathbf{1}(R_t \leq b + h^2)}{\sum_{t=1}^{T} \mathbf{1}(R_t \leq b + h^2)} \quad (23)$$

where $h^2_g > 0$ is a bandwidth and $K^2_g(.)$ a kernel with the support $[-1, 1]$. Then we plug in those estimator in (15). For the estimation based on the second stage’s winning prices, note that the weight ratios do not play any role. On the contrary, for the estimation based on the first stage, then we should take into account the weight ratio in the way we build pseudo-valuations. For each observation, we have a different estimator for the equilibrium bid function which is now estimated by

$$\hat{\psi}_t(x) = x - S_t \cdot \frac{\int_{u=0}^{\hat{G}_{P_2}(x)} \frac{[F^2(u)]^{2n-3} du}{[F^2(x)]^{2n-3}}}{\hat{G}_{P_2}(x) - \hat{G}_{P_2}(x)}$$

The way we build the pseudo-valuations becomes $X^1_t = \hat{\psi}_t^{-1}(B^1_t)$ if $B^1_t > R_t$ and $X^1_t = -1$ otherwise. For the estimation of $F(x)$ and $f(x)$ from the first stage bids, we plug in those pseudo-valuations in (21) and (22).

**Remark 5.4** Since we have a bunch of auctions with a null reserve price, we are able to estimate the full CDF $F$. E.g., the trimming to estimate the density $g_{P_2}$ is not a problem for recovering entirely the density $f$ because the lower bound $x$ of the distribution lies strictly above 0.

An important practical difficulty if one follows the methodology proposed above to complete our estimation procedure when auctions are heterogenous is that the estimator $\hat{G}_{P_2}$ may fail to be increasing and thus also our ‘direct’ estimator for $\hat{\psi}_t$ if no regularization is made. Note that those difficulties concern only the part of the estimation method based on the first stage’s winning price where the strict monotonicity of $\hat{\psi}$ is required for the definition of our estimator.

We thus let $\hat{G}_{P_2}(b) = \sup_{u \leq b} \hat{G}_{P_2}(b)$ [resp. $\hat{G}_{P_2}(b) = \inf_{u \geq b} \hat{G}_{P_2}(b)$] denote the lowest [resp. highest] CDF that lies above [resp. below] $\hat{G}_{P_2}$. Instead of taking $F^2(x) = \Psi^{-1}_{M_1}(\hat{G}_{P_2}(x))$, we consider the estimator

$$\hat{F}^2(x) = \Psi^{-1}_{M_1}(\frac{\hat{G}_{P_2}(x) + \hat{G}_{P_2}(x)}{2}) \quad (24)$$
in order to define $\hat{\psi}_i$ which is thus guaranteed to be increasing. This regularization procedure is illustrated in Figure 2 where the (preliminary) estimators $\Psi^{-1}_{\hat{M}_1}(\hat{G}_P(x))$, $\Psi^{-1}_{M_1}(G_P(x))$ and $\Psi^{-1}_{M_1}(G_P(x))$ are depicted for the data set we analyze in Section 7.

6 Monte Carlo Study

This section describes results of our Monte Carlo study when $n = 2$, $s = 1$, $r = 0$ and when the underlying marginal distribution $F$ that generates the data is the uniform distribution on $[0,1]$. We assume that buyers are playing the equilibrium following heuristic M1. We study the small sample properties of our estimation procedure and in particular the gain from using the winning prices at both stages.

Our finite sample distributions are based on 2000 replications for a sample size of $T = 100$. The bandwidths and kernels are chosen in the same way as in Bendstrup (2007): kernels are given by $K(x) = \frac{3}{4}(1 - x^2)$ for $x \in [-1, 1]$ and 0 otherwise; bandwidths are given by $0.79 \cdot \tau \cdot T^{-1/5}$ where $\tau$ is the interquantile range of the underlying data whose density is estimated. The weighting matrix $W$ and $\Omega$ are chosen to be the identity matrix. In Figure 3, the red curves correspond to the true CDF or PDF while the blue curves summarize our Monte Carlo simulations as indicated in the legend. Our estimator is not biased except at the bounds of the support: the problems in those areas come from the non-differentiability of the function $\Psi_{M_1}$ at the bounds, i.e. $\psi'_{M_1}(0) = \psi'_{M_1}(1) = 0$, a property that holds for any number of buyers. For the CDF, we note that this finite sample size bias is
Figure 3: The median, the 2.5, 10, 90 and 97.5 percentiles of $\hat{F}$ (Fig. 3a) and $\hat{f}$ (Fig. 3b) with our estimator with respect to the (correct) bidding heuristic M1 are depicted in blue. The true distribution is in red.

Figure 4: CDF of the mean squared error (MISE) of various estimators.

important only for the upper end of the support. For the PDF, there is also a bias due to a lack of boundary correction. This is the reason why we report only results on the interval $[0.1, 0.9]$. As it is commonly the case with nonparametric estimation, the variance for our estimator for the PDF is much larger than for the estimator of the CDF.

In Figure 4, we report the CDF of the mean squared error (MISE), $MISE = \int_0^1 (\hat{F}(x) - F(x))^2 \, dx$, of three estimation procedures when the data is generated from the equilibrium that is consistent with heuristic M1: first, in red, Bendstrup’s (2007) original estimator $\hat{F}(x) = \psi^{-1}_{R0}(\hat{G}_{P2}(x))$, second, in blue, the analog of Bendstrup’s (2007) estimator that uses only the last stage bids $\hat{F}(x) = \psi^{-1}_{M1}(\hat{G}_{P2}(x))$ and third, in black, our estimator that uses the bids from both stages. More precisely, we consider
a trimmed version of the MISE where the integral is on the support $[0.2, 0.8]$ to avoid the nuisances that occur at the bounds of the support. Naturally, our two estimators that are consistent with heuristic M1 clearly outperform the one that is consistent with heuristic R$_0$ that is strongly biased. More outstanding is the gain when we move to the estimation procedure that uses the winning price only at the last stage to the one that uses the winning price at both stages.

7 Application: tobacco auctions

In this section, we apply the methods derived above to a particular set of two-stage sequential English auctions held by a tobacco warehouse located in the southeastern United States. The data set, which corresponds to the one analyzed by Austin and Katzman (2002), henceforth A&K, consists in 90 twice-repeated auctions between August 2 and October 31, 1994. For each auction, the following information is available: the day at which each lot was auctioned and a ticket number capturing the order of the sales, a grade made by a government official which groups together three kinds of characteristics; the weight (in pounds) of each lot, the reserve price (on a per pound basis), the identity of the winner (if any) for each lot, the price (on a per pound basis) paid by the winner (if any) for each lot and whether each lot has been sold.

The grade corresponds to a set of discrete covariates. We are then following A&K by considering that only three binary variables are relevant to characterize the distribution of latent valuations. Those variables are then represented by a vector $Z \in \{\{0, 1\}\}^3$. Let $F_{X|Z}(\cdot|z)$ denote the underlying marginal distribution that characterizes the sampling scheme under the ID-sampling restriction. We then assume first that $F_{X|Z}(\cdot|z)$ depends on the realization $z$ of $Z$ only through an index $e^{\beta \cdot z}$ where the vector $\beta \in \mathbb{R}^3$ is unknown and has thus to be estimated, and second

---

29 Those three characteristics are the use, the color and the quality of the tobacco. See A&K for more details on those covariates. It is worthwhile to note that none of those auctions have exactly the same set of characteristics so that buyers can not ‘substitute’ between auctions that have exactly the same characteristics.

30 We emphasize that the reserve price is the same for both lots, a crucial feature that is necessary to apply our methodology. However, from the equilibrium analysis in Lamy (2012), we could easily adapt our identification and estimation analysis if the reserve price at the second stage were higher than the one at the first stage.

31 We could have included the weight of each lot as extra covariates. We do not since we found no correlation between weights and both winning prices or reserve prices.
that the influence of the index is multiplicative, i.e. \( F_{X|Z}(x|z) = F\left(\frac{x}{ez}\right) \) where \( F \) is the CDF of the normalized valuation we wish to estimate. Furthermore, we assume that the reserve price \( R \) is set so that

\[
R = e^{\beta Z} \cdot \epsilon
\]

where the noise \( \epsilon \) is an independent variable which satisfies \( \mathbb{E}[\log(\epsilon)] = 0 \). The vector \( \beta \) is then estimated from the linear regression \( \log(R) = \beta Z + \log(\epsilon) \).\(^{32}\) Let \( \hat{\beta} \) denote the OLS estimator. For each auction, we compute the variable \( e^{\beta Z} \cdot \epsilon \) which is called next the grade index.\(^{33}\) Table 2 provides the main summary statistics (the variable SOLD is an indicator being equal to one if the lot is sold and zero otherwise).

<table>
<thead>
<tr>
<th>Table 2: Summary statistics (full sample)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of observations</td>
</tr>
<tr>
<td>------------------------</td>
</tr>
<tr>
<td>Weight (in pounds)</td>
</tr>
<tr>
<td>Weight for the first lot</td>
</tr>
<tr>
<td>Weight for the second lot</td>
</tr>
<tr>
<td>Reserve price (per pound)</td>
</tr>
<tr>
<td>Reserve price if positive</td>
</tr>
<tr>
<td>SOLD</td>
</tr>
<tr>
<td>Price (per pound)</td>
</tr>
<tr>
<td>Grade index</td>
</tr>
</tbody>
</table>

From now on, we consider only normalized price data, i.e. the winning and reserve prices are normalized by their corresponding grade index. Furthermore, we also notice that the weight of the second lot is strictly larger than the one of the first lot in 39 auctions, i.e. in 43% of our sample. Because the assumption \( s \leq 1 \) is needed in Lamy’s (2012) equilibrium analysis,\(^{34}\) our subsequent analysis is thus

\(^{32}\)More precisely, we exclude from the regression the ten observations where the reserve has been set to zero. From our summary statistics, those reserves appear manifestly as outliers: the reserve price lies between 1.1 and 1.89 when it is positive. When the reserve price is null, then we suspect that the reserve price has not been fixed as a function of the covariates.

\(^{33}\)From Proposition 4.1, we could conjecture that the reserve price depends on the weight ratio and so in particular that the noise \( \epsilon \) is correlated with the weight ratio even if the distribution of latent valuations do not depend on lot sizes. If the grade index \( Z \) were correlated with the weight ratio, then this would introduce a bias in our estimator \( \hat{\beta} \). With an OLS regression, we can show that there is no significant correlation between those covariates. In any cases, the simulations in Section 4 suggest that the weight ratio has probably only a negligible impact on the optimal reserve price.

\(^{34}\)If \( s > 1 \), it can be shown that there is no equilibrium following heuristic M1, which prevents us definitely to apply our estimation procedure.
limited to the subsample of auctions with a weight ratio below one. Table 3 provides some summary statistics. We note that the weight ratio varies considerably across auctions ranging from 0.28 to 0.95. Eight percent of the lots remain unsold. The standard deviation of the reserve price is large (about one half of the mean) which is due to the significant proportion of auctions with no reserve price at all. On the contrary, if we limit ourselves to the subsample of auctions with positive reserve price, then the standard deviation of the reserve price is of the same order of that of the winning bids.

Table 3: Summary statistics (restricted sample with \( s \leq 1 \) and with normalized variables)

<table>
<thead>
<tr>
<th></th>
<th>Number of observations</th>
<th>Mean</th>
<th>Std. Dev.</th>
<th>Min</th>
<th>Max</th>
</tr>
</thead>
<tbody>
<tr>
<td>Reserve price</td>
<td>51</td>
<td>0.824</td>
<td>0.40</td>
<td>0</td>
<td>1.15</td>
</tr>
<tr>
<td>Reserve price if positive</td>
<td>42</td>
<td>1.001</td>
<td>0.096</td>
<td>0.80</td>
<td>1.15</td>
</tr>
<tr>
<td>Price for the first lot</td>
<td>47</td>
<td>1.027</td>
<td>0.13</td>
<td>0.59</td>
<td>1.24</td>
</tr>
<tr>
<td>if it is strictly above the reserve</td>
<td>45</td>
<td>1.028</td>
<td>0.14</td>
<td>0.59</td>
<td>1.24</td>
</tr>
<tr>
<td>Price for the second lot</td>
<td>47</td>
<td>1.037</td>
<td>0.12</td>
<td>0.60</td>
<td>1.24</td>
</tr>
<tr>
<td>if it is strictly above the reserve</td>
<td>44</td>
<td>1.039</td>
<td>0.12</td>
<td>0.60</td>
<td>1.24</td>
</tr>
<tr>
<td>SOLD (first lot)</td>
<td>51</td>
<td>0.92</td>
<td>0.27</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>SOLD (second lot)</td>
<td>51</td>
<td>0.92</td>
<td>0.27</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>weight ratio</td>
<td>51</td>
<td>0.80</td>
<td>0.15</td>
<td>0.28</td>
<td>0.95</td>
</tr>
</tbody>
</table>

Among the 102 lots we now consider, eight lots remain unsold while the other ones are bought by five different buyers. The total number of lots won by the five buyers are 29, 20, 19, 17 and 9. On the whole our symmetry assumption does not seem unrealistic. Next we do not take into account the information contained in the identity of the winners and consider that there are five symmetric potential participants.

Let us now comment shortly why Lamy’s (2012) setup fits the environment. First, there is no buyer’s option. Second, the set of participants is exogenous and fixed.

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\(^{35}\)In the full sample, the largest weight ratio equals 2.04.

\(^{36}\)In the full sample (that corresponds to the sample analyzed by A&K who do not take into account for the weight ratio and its strategic consequences in their analysis), there is an additional buyer who has won only two lots, and more precisely the two lots of a single auction. Contrary to A&K, we discard this additional buyer in our structural approach where we consider that there are only five potential buyers. According to private communication with Brett Katzman, the six buyers were on equal footing in this market and the fact that one of the six possible participants was a winner in only one auction is an anomaly. We thus implicitly assume that this sixth buyer is absent in the two month period covered by our data set and that the other buyers do take this into account in their bidding function at the first stage.
Third, though the sequential auctions are not always run immediately one after the other, the model seems appropriate. On the one hand, the two lots are auctioned exactly the same day in 73% of our sample, while the lag is typically of few days in the remaining cases. On the other hand, buyers know several weeks in advance the lots that will be auctioned, in particular once the lots (which are supplied by local farmers) arrive at the warehouse and are then graded by a government official. Buyers are also aware of lot sizes, and so of the weight ratio, well in advance of the auctions. Last, we think that the grade process makes the private value assumption especially relevant.

7.1 Basic tests

We begin with an examination of the individual observations in the data set. For each auction, there are nine possible outcomes depending on whether each lot is sold at the reserve price, sold for a price greater than the reserve price or remains unsold. From Corollary 2.2, the model allows for only five of these outcomes. That is, observation of any of the other four lead to an immediate rejection of the model. Among the 51 price sequences we have, we observe four different outcomes: both units remain unsold in four auctions, both units are sold at the reserve price in two auctions, the first lot is sold at a price greater than the reserve price while the second lot is sold at the reserve price in one auction, and both units are sold at a price greater than the reserve price in the remaining 44 auctions. On the whole, we are not able to reject the model based upon any single observation.

Regarding price trends, we find that the overall price sequence increases in expectation which is consistent with Proposition 2.3 and contrasts with the supposed declining price anomaly in sequential auctions, but the difference is not significant.

We also regress the reserve price and the price at the first stage on the weight ratio, but we found no statistically significant correlations. It should be noted that the difficulty to find significant results may have two causes here: on the one hand, the

---

37 A&K do the same exercise while taking into account the identity of the winner at each stage which leads to more possible outcomes and finer restrictions. Apart from the issue that they consider the full sample and thus include the auctions with a weight ratio larger than one for which there is a lack of theoretical predictions, they are also unable to reject the model based upon any single observation.

38 See Lamy (2012) for a discussion on the gap between the theoretical predictions and the predictions that have usually been tested in empirical studies.

39 See Table 3. With a two-sample t-test, we obtain that the corresponding p-value is 0.68.
Figure 5: Estimation of the marginal distribution $F$ (Fig. 5a) and density $f$ (Fig. 5b) for the valuation of tobacco: estimators from the last stage $\hat{F}_2$ and $\hat{f}_2$ (in red), from the first stage $\hat{F}_2$ and $\hat{f}_2$ (in blue) and from both stage $\hat{F}$ and $\hat{f}$ (in black).

small sample size, but also the fact that the number of buyers is large so that the “multi-unit demand” effects may be quite limited.

7.2 Estimations and counterfactual analysis

In order to estimate the demand for tobacco, we are following the methodology presented in Section 5 where kernels and bandwidths are chosen as in Section 6. The estimated CDF $\hat{F}$ [resp. PDF $\hat{f}$] is given in Figure 5a [resp. Figure 5b]. More precisely, we report the estimator from both stages but also from each stage separately. From visual inspection, it is striking that the estimators from the first and second stages separately are very close to each other for values above 0.9 which is consistent with our ID-sampling assumption. For the lower tail of the distribution, the noise comes for the fact that we rely on a very small sample: basically we are relying on the nine auctions for which there is no reserve price, among which only five lots have been sold at a price below 0.9.\footnote{Figure 2 does not solely illustrate our regularization procedure but also gives a flavor of the importance of the noise associated to our small sample size.}

In Figure 6, we present our estimate $\hat{\psi}(.)$ of the first-stage bidding function from Eq. (16) and also the analog where $\hat{F}_2$ has been replaced by $\hat{F}$ in (16).

Once we have estimated the marginal distribution of latent valuations, we have all the ingredients to run the counterfactual exercises proposed in Section 4. In what we report, we consider three different values for the seller’s valuation $X_S$: 0, 0.6
Figure 6: Estimation of the bidding function $\psi$ from the last-stage (red) and from both stages (black) for $s = 0.8$ and $r = 0.6$.

and $0.8$. $X_S = 0$ is the lower bound where the lot have zero value for the seller. $X_S = 0.6$ corresponds roughly to the case where the seller’s valuation equals to the lower bound of the valuation distribution (the lowest price for the second lot equals precisely 0.6). The status of $X_S = 0.8$ is that according to our calculation, it leads to an optimal reserve price which corresponds roughly to the mean of the reserve that are used in practice (when it is positive). In Table 4, we consider also three different values for the weight ratio ($s = 0.25, 0.8, 1$). In order to estimate the optimal reserve prices $r^*$ and $r^*_{\text{bund}}$, we use the empirical counterpart of (10) and (13), respectively. Then we report the expected payoff and the expected welfare in four counterfactual scenarios: the two-stage sequential auction with the optimal estimated reserve price $\hat{r}^*$, the welfare-maximizing two-stage sequential auction (i.e. with the reserve price $X_S$), the auction with bundling with the optimal estimated reserve price $\hat{r}^*_{\text{bund}}$ and finally the welfare-maximizing auction with bundling (i.e. with the reserve price $X_S$). Table 5 is devoted to counterfactual analysis with alternative number of potential buyers.

We first note that the expected payoff (which depends solely on the estimated CDF $\hat{F}$) is sometimes slightly larger when the reserve price is set at the seller’s valuation instead of our estimated optimal reserve price (this occurs for the counterfactuals with $X_S = 0$). This comes from the noise associated to the kernel estimation of the underlying density.\footnote{Our estimated reserve price depends not solely on the estimated CDF $\hat{F}$, but also on the estimated PDF $\hat{f}$.} When the number of buyers is large, then it is well-known
that the payoff function is flat below the optimal reserve price. Consequently, due to sample noise, the estimated optimal reserve price may be strictly higher than the optimal reserve and thus be worst than a much lower reserve as the seller’s valuation whose suboptimality has only negligible effects. In this vein and for English auctions, Aryal and Kim (2012) report some simulations which illustrate how estimated reserve prices can perform poorly. On the contrary, this effect should be less important when the number of buyers becomes smaller (see Table 5 for an illustration).

In all scenarios, we find that bundling in detrimental in terms of seller’s revenue. Furthermore, the difference is far from being negligible: when \( X_S = 0.6, \ s = 1, \ n = 5 \), the estimated revenue is 16% higher with sequential auctions compared to a single auction with bundling. Those differences are remarkable compared the extent of the impact of the reserve price policy. Furthermore, we find the same results when the number of buyers is lower, in particular for \( n = 2 \). Nevertheless, when \( n = 2, 3 \), we find that the impact of picking the optimal reserve is then at least as important as the bundling policy.

Concerning the optimal reserves, we find a complete different pattern compared to the simulations reported in Section 4: in all our simulations except in the two last columns of Table 4, we find that the optimal reserve is smaller in sequential auctions. Consequently, it is without surprise that we obtain that bundling is detrimental to the welfare: on the one hand, if the reserve is set at the seller’s valuation, then bundling imposes a constraint on the final possible assignment that is detrimental to the welfare; on the other hand, if the seller chooses the estimated optimal reserve, then the seller sets a higher reserve in auctions with bundling which prevents extra profitable sales.

Contrasting with the empirical literature that has consistently proposed optimal reserve prices that are much higher than actual reserve prices (see Paasch 1997, Li et al. 2003 and Haile and Tamer 2003), we find for this specific application that actual reserve prices are consistent with our estimation of the optimal reserve price.
Table 4: Counterfactual analysis with $n = 5$

<table>
<thead>
<tr>
<th>$X_S$</th>
<th>0</th>
<th>0.6</th>
<th>0.8</th>
</tr>
</thead>
<tbody>
<tr>
<td>s</td>
<td>0.25</td>
<td>0.8</td>
<td>1</td>
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Two-stage sequential auctions

<table>
<thead>
<tr>
<th></th>
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<th>$\hat{r}^*$</th>
<th>$\hat{r}^*$</th>
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</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.830</td>
<td>0.829</td>
<td>0.829</td>
<td>0.899</td>
<td>0.898</td>
<td>0.898</td>
</tr>
<tr>
<td>payoff if $r = \hat{r}^*$</td>
<td>1.233</td>
<td>1.658</td>
<td>1.812</td>
<td>1.269</td>
<td>1.739</td>
<td>1.910</td>
</tr>
<tr>
<td>payoff if $r = X_S$</td>
<td>1.233</td>
<td>1.662</td>
<td>1.818</td>
<td>1.233</td>
<td>1.662</td>
<td>1.818</td>
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<tr>
<td>welfare if $r = \hat{r}^*$</td>
<td>1.403</td>
<td>1.945</td>
<td>2.141</td>
<td>1.423</td>
<td>1.991</td>
<td>2.198</td>
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<td>welfare if $r = X_S$</td>
<td>1.432</td>
<td>2.011</td>
<td>2.221</td>
<td>1.432</td>
<td>2.011</td>
<td>2.221</td>
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</table>

A unique auction with bundling

<table>
<thead>
<tr>
<th></th>
<th>$\hat{r}^*_{\text{bund}}$</th>
<th>$\hat{r}^*_{\text{bund}}$</th>
<th>$\hat{r}^*_{\text{bund}}$</th>
<th>$\hat{r}^*_{\text{bund}}$</th>
<th>$\hat{r}^*_{\text{bund}}$</th>
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<tbody>
<tr>
<td></td>
<td>0.915</td>
<td>0.871</td>
<td>0.860</td>
<td>1.034</td>
<td>0.950</td>
<td>0.925</td>
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<td>payoff if $r = \hat{r}^*_{\text{bund}}$</td>
<td>1.173</td>
<td>1.459</td>
<td>1.561</td>
<td>1.217</td>
<td>1.529</td>
<td>1.644</td>
</tr>
<tr>
<td>payoff if $r = X_S$</td>
<td>1.210</td>
<td>1.583</td>
<td>1.718</td>
<td>1.210</td>
<td>1.583</td>
<td>1.718</td>
</tr>
<tr>
<td>welfare if $r = \hat{r}^*_{\text{bund}}$</td>
<td>1.289</td>
<td>1.618</td>
<td>1.738</td>
<td>1.282</td>
<td>1.636</td>
<td>1.772</td>
</tr>
<tr>
<td>welfare if $r = X_S$</td>
<td>1.362</td>
<td>1.800</td>
<td>1.962</td>
<td>1.362</td>
<td>1.800</td>
<td>1.962</td>
</tr>
</tbody>
</table>

Table 5: Counterfactual analysis with $s = 0.8$ and $X_S = 0.6$

<table>
<thead>
<tr>
<th>n</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
</table>

Two-stage sequential auctions

<table>
<thead>
<tr>
<th></th>
<th>$\hat{r}^*$</th>
<th>$\hat{r}^*$</th>
<th>$\hat{r}^*$</th>
<th>$\hat{r}^*$</th>
</tr>
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<tbody>
<tr>
<td></td>
<td>0.895</td>
<td>0.897</td>
<td>0.898</td>
<td>0.898</td>
</tr>
<tr>
<td>payoff if $r = \hat{r}^*$</td>
<td>1.412</td>
<td>1.543</td>
<td>1.650</td>
<td>1.739</td>
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<tr>
<td>payoff if $r = X_S$</td>
<td>1.195</td>
<td>1.371</td>
<td>1.530</td>
<td>1.662</td>
</tr>
<tr>
<td>welfare if $r = \hat{r}^*$</td>
<td>1.616</td>
<td>1.789</td>
<td>1.909</td>
<td>1.991</td>
</tr>
<tr>
<td>welfare if $r = X_S$</td>
<td>1.683</td>
<td>1.838</td>
<td>1.941</td>
<td>2.011</td>
</tr>
<tr>
<td>prob sale if $r = \hat{r}^*$</td>
<td>0.695</td>
<td>0.813</td>
<td>0.892</td>
<td>0.939</td>
</tr>
<tr>
<td>prob sale if $r = X_S$</td>
<td>0.972</td>
<td>0.997</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

A unique auction with bundling

<table>
<thead>
<tr>
<th></th>
<th>$\hat{r}^*_{\text{bund}}$</th>
<th>$\hat{r}^*_{\text{bund}}$</th>
<th>$\hat{r}^*_{\text{bund}}$</th>
<th>$\hat{r}^*_{\text{bund}}$</th>
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<tbody>
<tr>
<td></td>
<td>0.950</td>
<td>0.950</td>
<td>0.950</td>
<td>0.950</td>
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<tr>
<td>payoff if $r = \hat{r}^*_{\text{bund}}$</td>
<td>1.311</td>
<td>1.397</td>
<td>1.469</td>
<td>1.529</td>
</tr>
<tr>
<td>payoff if $r = X_S$</td>
<td>1.273</td>
<td>1.413</td>
<td>1.512</td>
<td>1.583</td>
</tr>
<tr>
<td>welfare if $r = \hat{r}^*_{\text{bund}}$</td>
<td>1.368</td>
<td>1.475</td>
<td>1.563</td>
<td>1.636</td>
</tr>
<tr>
<td>welfare if $r = X_S$</td>
<td>1.601</td>
<td>1.695</td>
<td>1.757</td>
<td>1.800</td>
</tr>
<tr>
<td>prob sale if $r = \hat{r}^*_{\text{bund}}$</td>
<td>0.362</td>
<td>0.490</td>
<td>0.593</td>
<td>0.675</td>
</tr>
<tr>
<td>prob sale if $r = X_S$</td>
<td>0.995</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

Remark 7.1 In sequential auctions, we have restricted ourselves to reserve price
policies where the reserve is constant over the sequences of stages within the auction while we could think of a wider class of reserve price policies. The theoretical analysis in Lamy (2012) encompasses also the cases where the reserve price in the second stage is higher than the one used in the first stage. Consequentially, we could also run counterfactuals with such a wider class of reserve price policies.

8 Conclusion

B&BP claim a very strong identification result for buyers’ valuations only from the last stage winning price distribution and without any assumption on the form of the information asymmetry, risk aversion and also whether agents are bidding according to some equilibrium criterion in the first stages of the auction sequence. On the contrary, we show that we definitely need such assumptions to identify the model and that non-identification might even occur if we model incomplete information in a standard way, and if we assume risk neutrality and that buyers are playing a Bayes Nash equilibrium. Since we need to rely on an equilibrium analysis, we then develop a structural approach for sequential auctions with multi-unit demands in the setup analyzed by Lamy (2012) where equilibrium uniqueness has been obtained under mild restrictions that are typically satisfied in practice. While it extends B&BP’s model in some directions (in particular the sampling scheme for buyers’ valuations), our analysis has important restriction with respect to B&BP’s ambitions: it is limited to two-stage instead of general multi-stage sequential auctions, English auctions are modeled as second-price auctions and finally buyers are symmetric ex ante.

First, two-stage sequential auctions may represent a large part of sequential auctions (e.g. in the data set used by B&BP, they represent 66% of the sample of sequential auctions) so that two-stage sequential auctions have an interest on their own.

Second, alternative models for English auctions may modify our analysis. With different modeling choices for the English auction, in particular if we consider that exits are irrevocable and publicly observed, then the equilibrium analysis would be slightly different as discussed in Lamy (2012). If $s = 1$, then it would reopen the multiplicity issue that occurs with only two potential bidders: the intuition is that the last phase of the English auction (i.e. when there are only two active bidders and
that it is publicly known) is roughly identical to a second-price auction with only two bidders. On the contrary, with \( s < 1 \), equilibrium behavior yields to a unique outcome and more precisely any equilibrium guarantees that the first lot is assigned to the bidder with the highest first valuation exactly as in our previous analysis. This is enough to be able to identify the model with the ID-sampling structure. However, our estimation methodology should be slightly adapted, and more precisely the part of the procedure that is based on first stage’s bids since the bidding function at the first stage is not the same as with sequential second-price auctions.\(^{42}\) A robust method consists in taking our estimator from the second stage’s winning prices.

Finally, we point out that we could deal with some form of asymmetry provided that bidders are still bidding symmetrically at the first stage, i.e. that heuristic M1 still hold. At first glance, this stand in contradiction with any form of asymmetry. However, if we have in mind a model with stochastic entry and with large population of entrants, then while being possibly asymmetric, buyers would face approximatively the same competition and thus use the same bid function in equilibrium. More generally with a large number of potential buyers, then assuming that bidders use the same bid function may be a reasonable approximation (see Fibich and Gavious 2003).

References


\(^{42}\)In particular, the bidding function at the last phase of the English auction when there are only two active bidders depends crucially on the bidding amount of the previous exit which reveals the valuation of the bidder with the third highest first valuation.


A Appendix

Proof of Proposition 3.1

We show that the derivatives of the polynomials $\Psi_{R_0}$ and $\Psi_{M_1}$ are strictly positive on $(0,1)$, which will guarantee that $\Psi_{i}^{-1}$ is differentiable on $(0,1)$. For $\Psi_{R_0}$, this has been already proved by Brendstrup (2007). Furthermore, we have $\Psi_{R_1}(x) = 2n(n-1)x^{2n-3}(1-x^2)$ which is strictly positive on $(0,1)$. Finally, we are done with $\Psi_{R_0}$. We now consider heuristic $M_1$ and work first conditional on $u$ the highest high valuation among all bidders. From Eq. (5) and since $f(x) > 0$ for any $x$ on the interval $[x, \bar{x}]$, we have $\tilde{g}_{P_2}(x|u) > 0$ for any $x \in (x, u)$. Since the density of the variable $u$ is strictly positive on $(x, \bar{x})$, we obtain then after the integration with respect to $u$ that $g_{P_2}(x) > 0$ for any $x$ on the interval $(x, \bar{x})$. Since $g_{P_2}(x) = \Psi_{M_1}(F(x)) \cdot f(x)$, we obtain finally that $\Psi_{M_1}'(x) > 0$ on $(0,1)$.

A straightforward factorization leads to $\Psi_{M_1}[X] - \Psi_{R_0}[X] = \frac{2(n-1)(2n-3)}{3}X^{2n-3}[1-X]^2 \cdot \left[ \frac{n-3}{2n-3} + X \right]$. Between two roots, a polynomial has a constant sign. The root $-\frac{n-3}{2n-3} \notin (0,1)$. A similar factorization leads to $\Psi_{M_1}[X] - \Psi_{R_1}[X] = \frac{2n(n-1)}{3}X^{2n-3}[1-X]^2 \cdot [1+2X]$. We obtain finally that $\Psi_{R_1}[X] < \Psi_{M_1}[X] < \Psi_{R_0}[X]$ on $(0,1)$ for $n = 2$ while $\Psi_{M_1}[X] - \Psi_{R_0}[X] > 0$ on $(0,1)$ for $n \geq 3$ and any $\rho \in [0,1]$. 


Proof of Proposition 4.1
For any \( x < x < \bar{x} \), we have
\[
\frac{dA(x)}{dx} = \int_{x}^{\bar{x}} \left[ f^2(x|u) - (1 - F^2(x|u)) \frac{f^1(x)}{1 - F^1(x)} \right] \frac{dF^1(u)}{1 - F^1(x)} > 0
\]
where the inequality comes from A10. In order to obtain the comparative static results, it is sufficient to show that \( \frac{\partial \xi(x)}{\partial s} > 0 \) and \( \frac{\partial \xi(x)}{\partial n} < 0 \) for any \( x < x < \bar{x} \).
From (10) and (11), those inequalities are equivalent to \( \frac{dA(x)}{dx} > 0 \) which concludes the proof.

B Calculations for online publication

B.1 Calculation of the optimal reserve price in sequential auctions

Since auctions that assign the two units in the same way are revenue equivalent, then \( \Pi(r) \) corresponds also to the expected revenue in the (generalized) Vickrey auction with the reserve price \( r \) per unit, i.e. the pivotal mechanism where the opportunity cost per unit for the seller equals \( r \).

Let \( F^{1(n)}(\cdot) \) denote the CDF of the \( i \)th order statistic among the first valuations of the \( n \) buyers. E.g., we have \( F^{1(2:n)}(x) := n[F^1(x)]^{n-1} - (n - 1)[F^1(x)]^n \). For any pair \((u, z)\) with \( z \geq u \), let \( Q(\cdot|u, z) \) denote the CDF of the highest valuation among the first valuations of \( n - 2 \) buyers given that they have first valuations below \( u \) and the second valuation of a buyer given that his first valuation equals \( z \) and that his second valuation is below \( u \). By independence, we have \( Q(\cdot|u, z) = \frac{F^1(\cdot)}{F^1(\cdot) - F^2(\cdot|z)} \frac{F^1(z)^{n-2} F^2(\cdot|z)}{F^2(\cdot|z)} \). Let \( \pi_Q(u, z) \) denote the expected value of the maximum of this latter variable and the reserve price \( r \), i.e. \( \pi_Q(u, z) := \int_{0}^{u} (y \vee r) dG(y|u, z) \). Note that \( \pi_Q(r, z) = r \) and \( \frac{d\pi_Q(u, z)}{dr} = G(u|u, z) - G(r|u, z) \) for \( u > r \). From a tedious but straightforward calculation, we obtain that

\[43\] The Ausubel’s (2004) auction implements the (generalized) Vickrey auction when \( s = 1 \). It can be easily extended to slightly heterogenous units, i.e. when \( s \neq 1 \).
\[
\Pi(r) = (1 + s)X_S \cdot F_1^{(1:n)}(r) \\
+ ((1 + s)r + s(X_S - r) \cdot A(r)) \cdot [F_1^{(2:n)}(r) - F_1^{(1:n)}(r)] \\
+ \int_r^\pi \int_u^r (u + s\pi_Q(u,u))[1 - F_2(u|z)] \cdot d\left[\frac{F_1(z)}{1 - F_1(u)}\right] \cdot d[F_1^{(2:n)}(u)] \\
+ \int_r^\pi \int_u^r [(1 - s)u + s\pi_Q(u,u) + s\pi_Q(u,z)]F_2(u|z) \cdot d\left[\frac{F_1(z)}{1 - F_1(u)}\right] \cdot d[F_1^{(2:n)}(u)] 
\]

where the first term corresponds to the cases where the seller keeps both units, the second term corresponds to the cases where one and only one buyer has a first valuation above \( r \), the third term corresponds to the cases where at least two buyers have a first valuation above \( r \) and where the two units are assigned to the same buyer which pays on average \( u + s\pi_Q(u,u) \) conditional on the first valuation of his highest opponent being equal to \( u \), and the fourth term corresponds to the cases where at least two buyers have a first valuation above \( r \) and where the two units are assigned to different buyers (given that those two buyers have the first valuations \( z \) and \( u \), respectively (with \( z \geq u \), then the buyer with the first valuation \( z \) [resp. \( u \)] pays \( (1 - s)u + s\pi_Q(u,u) \) [resp. \( s\pi_Q(u,z) \)] on average)).

Similarly, the expected welfare \( W(r) \) is given by:

\[
W(r) = (1 + s)X_S \cdot [F_1(r)]^n \\
+ \int_r^\pi \left( u + s \cdot (X_S F_2(r|u))\left[\frac{F_1(r)}{F_1(u)}\right]^{n-1} \right) \cdot d[F_1(u)]^{n}.
\]
\[
\frac{d\Pi(r)}{dr} = (1 + s)X_S \cdot n[F1(r)]^{n-1} f1(r) \\
+ ((1 + s) - sA(r) + s(X_S - r) \cdot \frac{dA(r)}{dr}) \cdot n[F1(r)]^{n-1}(1 - F1(r)) \\
+ ((1 + s)r + s \cdot (X_S - r) \cdot A(r)) \cdot n(n-1)[F1(r)]^{n-2}(1 - F1(r)) f1(r) \\
- ((1 + s)r + s \cdot (X_S - r) \cdot a(s)) \cdot n[F1(r)]^{n-1} f1(r) \\
- (1 + s)r \cdot n(n-1)[F1(r)]^{(n-2)}(1 - F1(r)) f1(r) \\
+ s \cdot \int_\pi^r \int_u \pi G(r|u, u) \cdot d\left[\frac{F1(z)}{1 - F1(u)}\right] d[F1(2:n)(u)]_{\equiv B(r)} \\
+ s \cdot \int_\pi^r \int_u \pi G(r|u, z) F2(u|z) \cdot d\left[\frac{F1(z)}{1 - F1(u)}\right] d[F1(2:n)(u)]_{\equiv C(r)}
\]

For \(r \in (\pi, \pi)\), we have then

\[
\frac{d\Pi(r)/dr}{n[F1(r)]^{n-1} f1(r)} = (1 + s)(X_S - r) \\
+ ((1 + s) - sA(r) + s(X_S - r) \cdot \frac{dA(r)}{dr}) \cdot \frac{(1 - F1(r))}{f1(r)} \\
+ s \cdot (X_S - r) \cdot A(r) \cdot (n-1) \cdot \frac{(1 - F1(r))}{F1(r)} \\
- s \cdot (X_S - r) \cdot A(r) \\
+ s \cdot \frac{(B(r) + C(r))}{n[F1(r)]^{n-1}(1 - F1(r))} \cdot \frac{(1 - F1(r))}{F1(r)}
\]

\[
\frac{B(r)}{n[F1(r)]^{n-1}(1 - F1(r))} = \int_\pi^r \frac{F1(u)}{F1(F1(u))^{n-2}F2(r|u) \cdot d[F1(2:n)(u)]}{n[F1(r)]^{n-1}(1 - F1(r))} \\
= (n - 1) \cdot \int_\pi^r F2(r|u) \frac{(1 - F1(u))}{F1(r)} \frac{d[F1(u)]}{1 - F1(r)}
\]
We can check that

\[
\frac{C(r)}{n[F1(r)]^{n-1}(1 - F1(r))} = \int_{r}^{\bar{r}} \int_{u}^{\bar{r}} \frac{F1(z)^{n-2} F2(r|z) \cdot d[F1^{(2n)}(u)][F1^{(2n)}(u)]}{n[F1(r)]^{n-1}(1 - F1(r))}
\]

\[
= (n - 1) \cdot \int_{r}^{\bar{r}} \int_{u}^{\bar{r}} \frac{F2(r|z) \cdot d[F1(u)]}{F1(r)} \frac{d[F1(u)]}{1 - F1(r)}
\]

\[
= (n - 1) \cdot \int_{r}^{\bar{r}} F2(r|u) \frac{(F1(u) - 1) \cdot d[F1(u)]}{1 - F1(r)}
\]

We have then

\[
\frac{B(r) + C(r)}{n[F1(r)]^{n-1}(1 - F1(r))} = (n - 1) \cdot \int_{r}^{\bar{r}} \frac{(F2(r|u))}{F1(r)} d[F1(u)] = A(r) \cdot (n - 1) \frac{(1 - F1(r))}{F1(r)}
\]

Finally, we obtain:

\[
\frac{dM(r)/dr}{n[F1(r)]^{n-1}F1(r)} = (X_S - r) \left[ 1 + s \cdot \left( 1 - A(r) + A(r) \cdot (n - 1) \frac{(1 - F1(r))}{F1(r)} + \frac{dA(r)}{dr} \frac{(1 - F1(r))}{f1(r)} \right) \right]
\]

\[
+ \left[ 1 + s \cdot \left( 1 - A(r) + A(r) \cdot (n - 1) \frac{(1 - F1(r))}{F1(r)} \right) \right] \cdot \frac{(1 - F1(r))}{f1(r)}
\]

\[
= \left[ 1 + s \cdot \left( 1 - A(r) + A(r) \cdot (n - 1) \frac{(1 - F1(r))}{F1(r)} + \frac{dA(r)}{dr} \frac{(1 - F1(r))}{f1(r)} \right) \right] \cdot (X_S - \xi(r))
\]

where \(\xi(.)\) is defined in (10). We can check easily that \(\frac{dA(r)}{dr} \frac{(1 - F1(r))}{f1(r)} \geq -1\) and that \(sA(r) \cdot [1 - (n - 1) \frac{(1 - F1(r))}{F1(r)}] < 1\) for any \(r < \bar{r}\), which implies then that \(E(r) > 0\).

On the whole, we have shown that under A9, \(\frac{dM(r)/dr}{n[F1(r)]^{n-1}F1(r)}\) is quasi-monotone increasing on \((\underline{r}, \bar{r})\) and thus that the optimal reserve \(r^*\) is then characterized by the first-order condition \(\frac{dM(r^*)}{dr} = 0\) or equivalently \(r^* = \xi^{-1}(X_S)\).

We can also check that \(\xi(r) \geq r\) for any \(r \in (\underline{r}, \bar{r})\) and we obtain thus as a corollary that \(r^* \in [X_S, \bar{r}]\).

**Remark B.1** We can check that \(\frac{dA(r)}{dr} = 0\) if buyers have unit-demand. Under this special case or if there is only one item for sale (i.e. \(s = 0\)), then the optimal reserve denoted by \(r_{u}^*\) satisfies the well-known formula \(r_{u}^* - X_S = \frac{1 - F1(r_{u}^*)}{f1(r_{u}^*)}\).

**Calculation under ID-sampling schemes**

Under A3b, we have \(Q(.|u, z) = Q(.|u, u) = \frac{F1(.)}{F(u)} \cdot 2^{n-3}\) so that \(\pi_Q(u, z) = \pi_Q(u, u)\).

We have also \(A(r) = \frac{2F(r)}{1 + F(r)}\). On the whole (25) simplifies then to
\[ \Pi(r) = (1 + s)X_S \cdot [F(r)]^{2n} + ((1 + s)r + s(X_S - r) \cdot A(r)) \cdot n[F(r)]^{2n-2}(1 - F^2(r)) + \int_r^x 2n(n-1)(u + s \pi Q(u, u))F^{2n-3}(u)(1 - F(u))^2 \cdot f(u)du + \int_r^x 4n(n-1)((1 - s)u + 2s \pi Q(u, u)) \cdot F^{2n-2}(u)(1 - F(u)) \cdot f(u)du \]

and the optimal reserve price is characterized by

\[ r^* - X_S = \frac{1 - F^1(r^*)}{f^1(r^*)} \left[ 1 + s \cdot \left( \frac{1 - F^1(r^*) + 2(n - 1) F^2(r^*)}{1 + F^1(r^*)} \right) \right] \]

\[ = \frac{1 - F^1(r^*)}{f^1(r^*)} \left[ 1 + s \cdot \left( \frac{2n - 1}{1 + F^1(r^*)} \right) \cdot \frac{1 - F^1(r^*)}{F^1(r^*)} \right]. \]

### B.2 Calculation of the optimal reserve price under bundling

The sellers’ expected payoff and the expected welfare under bundling as a function of the reserve price are given by

\[ \Pi_{\text{band}}(r) = (1 + s)X_S[F_{\text{band}}((1 + s)r)]^n + (1 + s)r \cdot n[F_{\text{band}}((1 + s)r)]^{n-1}(1 - F_{\text{band}}((1 + s)r)) + \int_{(1+s)r}^{(1+s)x} u \cdot n(n-1)[F_{\text{band}}(u)]^{n-2}(1 - F_{\text{band}}(u))dF_{\text{band}}(u) \]

and

\[ W_{\text{band}}(r) = (1 + s)X_S[F_{\text{band}}((1 + s)r)]^n + \int_{(1+s)r}^{(1+s)x} u \cdot d[F_{\text{band}}(u)]^n. \]

Under A9, we obtain that the optimal reserve price \( r^*_{\text{band}} \) is characterized by

\[ r^*_{\text{band}} - X_S = \frac{1 - F_{\text{band}}((1 + s)r^*_{\text{band}})}{(1 + s)f_{\text{band}}((1 + s)r^*_{\text{band}})}. \]

Under A3b, note that the expression of the CDF \( F_{\text{band}} \) and the PDF and \( F_{\text{band}} \)
simplifies to

\begin{align*}
F_{\text{bund}}(x) &= \left[F\left(\frac{x}{1+s}\right)\right]^2 + 2\int_{\frac{x}{1+s}}^{x} F\left(\frac{x-u}{s}\right)dF(u) \quad \text{and} \quad f_{\text{bund}}(x) = \frac{2}{s} \cdot \int_{\frac{x}{1+s}}^{x} f\left(\frac{x-u}{s}\right)f(u)du.
\end{align*}