Equilibria in two-stage sequential second-price auctions with multi-unit demands

Laurent Lamy*

Abstract

We characterize the set of symmetric pure strategy equilibria in sequential second-price auctions with multi-unit demands when two units of an homogenous good are put for sale and when valuations are drawn independently from a general bivariate density. There is a unique equilibrium when there are at least three bidders while a continuum of equilibria prevails with only two bidders. Equilibrium behavior produces efficient outcomes and an expectation of increasing prices. More generally, we consider also perturbations of the model where some extra bidders may enter the auction game at the second stage, which provides a selection criterium when equilibrium multiplicity prevails. Sequential English auctions are also considered. Our results have important implications for empirical studies.

Keywords: Sequential auctions, Multi-unit demand, Second-price auction, English auction, Multi-dimensional information, Multiple equilibria

JEL classification: D44, D82

1 Introduction

In the past two decades there has been a vast increase in the empirical literature on auctions. While the received theory primarily focus on the sale of single and indivisible objects, it is often the case in reality that a sale involves multiple units of the same good and that those units are sold sequentially. While the theory has been extended to accommodate sequential auctions, it has been mainly done under the strong assumption that potential buyers demand at most one unit. However, multi-unit demand is the norm rather than the exception in real-life auctions. Due to this void, empirical research typically involves reduced form approaches and principally relies on the analysis of price trends in the perspective of the theoretical predictions that have been derived with single-unit demands. Under multi-unit demand, equilibrium analysis are challenging, if not untractable

*Paris School of Economics, 48 Bd Jourdan 75014 Paris. e-mail: laurent.lamy@pse.ens.fr
in general, in particular due to ‘ratchet’ effects.\textsuperscript{1} Nevertheless, for two-stage sequential second-price auctions, Katzman \cite{25} develops a general model under incomplete information where he analyzes symmetric equilibria which involve so-called \textit{separable} bid functions, i.e. where bids at the first stage depend either exclusively on the marginal valuation for the first unit or exclusively on the marginal valuation for an additional second (and last) unit. He shows that there always exists an equilibrium where the bid function at the first stage is based solely on the marginal valuation for the first unit. When there are only two bidders, he also exhibits another equilibrium where first stage’s bids are equal to bidders’ marginal valuations for the second unit. The restriction to separable bid functions has no theoretical status and we could conjecture from Katzman’s \cite{25} analysis that the equilibrium multiplicity caveat is even larger if we were considering the set of symmetric pure strategy equilibria. Furthermore, equilibrium multiplicity is highly problematic if we want to develop structural approaches aiming to recover bidders’ valuation distribution.\textsuperscript{2} In particular, Lamy \cite{28} shows that Katzman’s \cite{25} model is not identified from the last stage price if there are only two bidders.\textsuperscript{3} The lack of knowledge of the set of equilibria in sequential auctions is also problematic for experimental works: the few studies that consider multi-unit demands rely on stylized setups with a very limited set of types so that the authors are able to characterize easily the set of equilibria.\textsuperscript{4} It is thus of crucial interest to be able to characterize the full set of equilibria in two-stage sequential second-price auctions with multi-unit demands, an auction game that is commonly used in practice.

Our contribution with respect to Katzman \cite{25} is severalfold. On the one hand, we provide a characterization of the set of symmetric equilibria in \textit{regular} pure strategies, where the regularity restriction is related to the smoothness of the bid function so that expected payoff functions are differentiable almost everywhere and that we can exploit first-order conditions. We strongly emphasize that we do not impose any monotonicity restriction. The properties that equilibrium behavior produces efficient outcomes, that prices follow a submartingale and that bidders are typically indifferent among a large set of bids hold for any equilibrium and thus not solely for the two notable equilibria exhibited by Katzman \cite{25}. Furthermore, we show that the equilibrium is unique if the number of

\textsuperscript{1} Models under complete information avoid the difficulties associated to such reputation effects. With two bidders, Gale and Stegeman \cite{18} characterize a unique equilibrium allocation for any number of stages when an arbitrary number of homogenous units are put for sale.

\textsuperscript{2} See Paarsch and Hong \cite{36} for a textbook on the structural econometrics of auction data.

\textsuperscript{3} This stands in contradiction with the identification claims in Brendstrup \cite{6} and Brendstrup and Paarsch \cite{7} where those authors consider a generalization of Katzman’s \cite{25} model for any number of stages and also the possibility of asymmetric bidders in Brendstrup and Paarsch \cite{7}. See Lamy \cite{28} for a detailed explanation of the flaws in those papers.

\textsuperscript{4} Février et al. \cite{16} and Cason et al. \cite{10} consider respectively two-stage sequential English auctions with three types and two-stage sequential first-price auctions with two types.
potential bidders is larger than three. With only two bidders, we show on the contrary that a continuum of equilibria prevails and we discuss equilibrium selection. On the other hand, our model is a generalization of Katzman’s [25] framework with several respect. We first consider a much larger class of distributions to model multi-unit demand insofar that the pair of valuations of each buyer is drawn from a general bivariate density instead of resulting from two independent draws from a common univariate density. Second, we are allowing for binding reserve prices, an extension which appears to be fruitful when we move to the analysis of sequential English auctions. Third, we consider the possibility of unrestricted additional competition at the second stage through extra bidders that may win this second auction or raise the price paid by the winner. As in Katzman [25], there is always an equilibrium bid function which is based solely on the marginal valuation for the first unit, say $v_1$, and where bids are equal to the probability of winning at the second stage times the corresponding expected price plus the probability of losing at the second stage times $v_1$, with all expectations being conditional on the event where the highest marginal valuation for the first unit among the given bidder’s opponents at the first stage is equal to $v_1$ (meaning that a tie has occurred). This bid function is the unique equilibrium not only if there are at least three bidders, but also if extra bidders may enter at second stage and possibly outbid any initial bidder. Finally, we also analyze briefly the case where bidders are risk-averse, extending then partially McAfee and Vincent’s [32] analysis from single-unit to multi-unit demands and in particular their result about the non-existence of efficient equilibria when bidders’ utility function displays decreasing absolute risk aversion.

As in Katzman [25], our analysis of sequential second-price auctions is limited to symmetric bidders, private values and two units for sale. At the second stage, each buyer has thus a weakly dominant strategy: it consists in bidding his marginal valuation for one more unit, i.e. the winning bidder at the first stage should bid his marginal valuation for a second unit while losing bidders should bid their marginal valuations for a first unit. From the first stage’s perspective, our analysis corresponds then to the one of a single second-price auction with symmetric bidders having bi-dimensional private signals and with both informational and allocative externalities. Beyond tractability issues, equilibrium existence (even in mixed strategies) is not guaranteed in models with interdependent values that involve multi-dimensional information (Jackson [22]). Nevertheless, the analysis remains tractable here since those externalities correspond to the continuation values resulting from a second-price auction under private values, a game which has a very specific structure.

This paper belongs more generally to the subfield of auction theory which deals with auctions with future interactions and where bidders’ continuation values do not depend
solely on their own private information and whether they get the good or not but may also depend on the types of their opponents and their corresponding beliefs on those types. In particular, bid disclosure rules may influence continuation values since bidders’ beliefs at future stages are shaped by what has been revealed throughout the auction process. Sequential auctions is one of the most prominent topic of interest in this subfield which covers also auctions with allocative externalities (Jehiel and Moldavouni [23]), with resale opportunities (Haile [21]) or with an aftermarket (Goeree [20]). Regarding this literature and auction theory more generally, our analysis is one of the few that characterizes the set of equilibria in a standard auction format and for a model involving bi-dimensional private information. To the best of our knowledge, the three notable exceptions are Gale and Hausch [17] in a sequential auction model for two heterogenous objects among two bidders having single-unit demand, Che and Gale [12] in a model where additional to their private valuation, bidders are informed about a parameter capturing their own cost for money, and de Frutos and Pechlivanos [14] in a model with interdependent valuations.

Though there are no formal connections, our results regarding equilibrium multiplicity are reminiscent to the ones obtained by Bikhchandani and Riley [4]. In a pure common value model with symmetric bidders, Bikhchandani and Riley [4] show that the set of equilibria in second-price auctions switches from a continuum to a singleton when we move from two bidders to more than two bidders. On the contrary, in English auctions (with observable dropout prices), equilibrium multiplicity reemerges since the last stage of the ascending auction, i.e. when they are only two active bidders, can be viewed as a second-price auction with two bidders and with modified initial conditions. This latter phenomenon reemerges also in our model stressing thus the importance of equilibrium selection criteria.

This paper is organized as follows. In Section 2, we present the model, the notation and the conditional expectations that are used then to express the payoff functions and to derive first-order conditions. Section 3 is the core of the paper: after technical preliminaries which consist in providing a series of necessary conditions satisfied by any equilibrium bid function, we characterize the set of symmetric pure strategy equilibria for two-stage sequential second-price auctions and we derive several properties satisfied by all equilibria. Uniqueness is obtained under mild conditions while equilibrium refinements are discussed in case of multiplicity. Sections 4 and 5 are devoted respectively to risk aversion and to English auctions. Section 6 concludes. Most proofs are relegated to Appendices A-V. Only the proofs that contain the most crucial ingredients are put in the body of the paper.
2 The model

We consider a generalization of the model under incomplete information analyzed by Katzman [25]. Two identical objects are sold to \( N \) symmetric risk-neutral buyers. A non-strategic auctioneer sells the two units using a sequence of second-price auctions with a common reserve price \( r \geq 0 \), which is also interpreted as the seller’s reservation value.\(^5\) In the case where there is a tie at the highest bid, the item is awarded with equal probability to one of the highest bidders. Each buyer’s (privately known) type \( v = (v_1, v_2) \in \mathbb{R}^2_+ \) is drawn independently from the publicly known bivariate atomless distribution \( F \) with the corresponding continuous density \( f > 0 \) on its support \( T \) which is a subset of \( \{(v_1, v_2) \in [0, \pi]^2 : v_1 \geq v_2 \} \) with \( \pi > r \) and a superset of \( T_r := \{(v_1, v_2) \in [0, \pi]^2 : v_1 \geq r \lor v_2 \}. \) Since buyers are assumed to be symmetric while we restrict our analysis to symmetric equilibria we drop the references to any particular buyer in the subsequent notation. The type \( v \) of a given buyer corresponds to a pair of valuations: \( v_1 \) represents his marginal valuation for a first unit (henceforth called briefly his first valuation) and \( v_2 \) his marginal valuation for a second unit (henceforth called his second valuation).\(^6\) Let \( \Delta(T) := \{E \subseteq T : \text{Prob}(E) = \int \int 1[v \in E] f(v) dv > 0 \} \) denote the set of the subsets of types that occur with strictly positive probability. For \( E \in \Delta(T) \) and \( i = 1, 2 \), let \( F_i(\cdot|E) \) denote the CDF of the \( i \)th valuation conditional on \( v \in E \): \( F_i(x|E) = \frac{\int \int \mathbb{1}[v_1 \leq x \land v \in E] f(v) dv}{\text{Prob}(E)} \). Let \( F_1(\cdot) := F(\cdot, \pi) \) and \( F_2(\cdot) := F(\pi, \cdot) \) denote the CDFs of respectively the first and second valuation, which are continuously differentiable on \([r, \pi]\). For \( v \in T_r \), let \( F_2(\cdot|v_1) \) denote the CDF of the second valuation conditional on the first valuation being equal to \( v_1 \): \( F_2(v_2|v_1) := \frac{\int_0^{v_2} f(v_2, u) du}{\int_0^{v_1} f(v_1, u) du} \) if \( v_1 > 0 \) and \( F_2(v_2|0) := \mathbb{1}[v_2 \geq 0] \). Note that \( v \to F_2(v_2|v_1) \) is continuous on \( T_r \). We also assume that for any \( u \in [0, \pi] \), the restriction of \( F_2(u|\cdot) \) on \([u \lor r, \pi]\) is continuously differentiable. The sampling scheme considered here to generate buyers’ valuations is much more general than the one used in Katzman [25] which assumes that the pair of valuations of a given buyer results from two independent draws from a common underlying univariate CDF.\(^7\) With respect to our model, Katzman [25] implicitly imposes the restriction that

\(^5\)This is just a practical interpretation for our underlying notion of efficiency. In our model, the seller is not strategic so that this specification does not play any role in our equilibrium analysis. We also emphasize that we assume implicitly that the auctioneer is able to commit at the beginning of the game to the reserve price in both auctions. Otherwise ‘ratchet’ effects would arise as in Caillaud and Mezzetti [9].

\(^6\)In the literature on sequential auctions, almost all theoretical papers have considered models with single-dimensional types. This is natural when bidders have single-unit demand and that the goods for sale are homogenous, but this is quite restrictive for more general setups (e.g., Black and de Meza [5] consider that the second valuation is a fixed share of the first valuation while Pitchik [38] captures the valuations for two heterogenous goods and also a budget-constraint through a single-dimensional type).

\(^7\)This special structure in Katzman’s [25] model is borrowed from Noussair [35] which analyzes uniform price auctions with two-unit demands (but with no restriction on the number of units for sale) and provides a characterization result after imposing strong monotonicity restrictions on the bid function.
\[ F_2(v_2|v_1) = \left[ \frac{F_1(v_2)}{F_1(v_1)} \right]^{1/2} \] for any \( v \in T_r \). Furthermore, we also slightly extend Katzman’s [25] model by allowing for some extra bidders arriving exogenously at the second stage and then competing with the previous buyers that are referred to as the initial buyers. This extension will be fruitful insofar that it offers a refinement criterium to deal with equilibrium multiplicity. This is captured by the right-continuous CDF \( F^{ex} \) which denotes the distribution of the highest bid (or equivalently the highest valuation) among those extra bidders. Let \( b_\text{ex} := \sup \{ b \in \mathbb{R} : F^{ex}(b) = 0 \} \) and \( \bar{b}^{ex} := \inf \{ b \in \mathbb{R} : F^{ex}(b) = 1 \} \) denote respectively the lowest and highest bound of the support of \( F^{ex} \). Without loss of generality for our equilibrium analysis, we assume that the valuations of extra bidders belong to the interval \([r, v]\), or equivalently that \( r \leq b_\text{ex} \leq b^{ex} \leq v \), and that there is actually only one extra bidder. If \( b^{ex} = r \), then we are back to Katzman’s [25] setup with no extra bidders. On the opposite, if \( b^{ex} = v \), then almost all types could be outbid with some positive probability by the extra bidder at the second stage. If \( b^{ex} = b_\text{ex} = r' \geq r \), then our model can also be interpreted as the extension where the reserve price at the second stage may be larger than the one at the first stage. We emphasize that we never allow for any kind of perturbation by extra bidders in the first auction. We let \( x^* := v \) if \( N \geq 3 \) and \( x^* := b^{ex} \) if \( N = 2 \).

Remark 2.1 The case where \( F^{ex}(b) = (1 - \alpha) \cdot 1[b \geq r] + \alpha \cdot 1[b \geq v] \) with \( \alpha \in [0, 1] \) receives two interesting interpretations in terms of extensions of the basic model with no extra bidders. It can be viewed either as the extension where the second auction is canceled with probability \( \alpha \geq 0 \) or as the extension where the two units are not exactly identical and more precisely where buyers’ value [resp. the reserve price] for the second unit for sale is a fixed share \((1 - \alpha) \leq 1 \) of their value [resp. the reserve price] for the first unit (e.g. because the second unit is of smaller size or of lower quality). This last interpretation may be of crucial importance empirically since many sequential auctions for slightly heterogenous goods may actually be viewed as auctions of bundles of different size (or quality) of an homogenous good.

---

8Throughout the paper we use the convention \( 0/0 = 1 \).
9Or equivalently if \( F^{ex}(b) = 1[b \geq r'] \) with \( r' \geq r \).
10Such perturbations would modify the analysis considerably and raise tractability issues (even if those extra bidders are not strategic). In particular, a model where the reserve price in the second auction is strictly lower than the one in the first auction opens the door for positive allocative externalities, and this arises even under single-unit demand. Furthermore, the existence of symmetric pure strategy equilibria is typically not guaranteed in presence of positive allocative externalities (see Haile [21] and Jehiel and Moldovanu [23]).
11The values of winning only the first unit, only the second unit and both units are then equal respectively to \( v_1, (1 - \alpha) \cdot v_1 \) and \( v_1 + (1 - \alpha) \cdot v_2 \) while the reserves are linked together by \( r_2 = (1 - \alpha) \cdot r_1 \) where \( r_i \) denotes the reserve price at the \( i \)th stage. We will explain later (in Appendix O) why our analysis does not extend when \((1 - \alpha) > 1\). It is actually related to the issue of positive externalities mentioned in footnote 10.
In the second (and last) auction, it is assumed that all bidders follow their weakly dominant strategy consisting in bidding their valuation for the remaining unit, i.e. the winning [resp. a losing] bidder in the first auction will bid his second valuation [resp. his first valuation] in the second auction. We are thus left with buyers’ strategies in the first auction. The strategy space of a buyer is denoted by \( B_r := \{-1\} \cup [r, \infty) \), where the bid \(-1\) corresponds to non-participation and \([r, \infty)\) to the set of active bids. Throughout the paper, we assume that buyers are using a common pure strategy or bid function which is denoted by \( \beta : T \rightarrow B_r \), i.e. \( \beta(v) \) corresponds to the bid submitted by a buyer if he has valuation \( v \in T \).

We now introduce some additional notation for a given equilibrium candidate \( \beta \). However, to alleviate the mathematical expressions, the dependence w.r.t. \( \beta \) will usually be dropped in our notation. Let \( P := \{ v \in T : \beta(v) \geq r \} \) denote the set of types that submit active bids in the first auction. We can check immediately that we have necessarily \( P \in \Delta(T) \) in equilibrium. Let \( G(.) \) denote the CDF of the bid of a given buyer at the first auction: \( G(b) := \int \mathbf{1}_{\beta(v) \leq b} f(v) dv \). Since \( G \) is monotonic, it is differentiable almost everywhere. Once properly defined the corresponding derivative is denoted by \( g(b) \). Let \( G(b) \) denote the limit from the left of \( G \) at \( b \) or equivalently \( G(b) := \int \mathbf{1}_{\beta(v) < b} f(v) dv \). We emphasize that we do not exclude that \( G \) may contain some atoms for some active bids, i.e. there may exist \( b \geq r \) such that \( G(b) - G(b) > 0 \).\(^{12}\) Let \( S^A_+ \) [resp. \( S^A \)] denote the set of the atoms of the distribution \( G \) that lie in \([r, \infty)\) [resp. \( B_r \)]. Let \( S := \{ b \in B_r : \exists v \in P \text{ such that } \beta(v) = b \} \) denote the set of active bids that are used in equilibrium. Let \( \underline{b} := \inf \{ b \in S \} \) and \( \bar{b} := \sup \{ b \in S \} \). Let \( S^* := \{ b \in S : G(b) > 0 \} \). Let \( S^{NA} \) denote the set of the bids \( b \in S^* \) such that \( G(.) \) is continuously differentiable at \( b \) and with \( g(b) > 0 \). We have \( S^{NA} \subseteq S^* \subseteq S \subseteq [\underline{b}, \bar{b}] \). We also use the notation \( F_i(\cdot|\beta \leq b) \), \( F_i(\cdot|\beta < b) \) and \( F_i(\cdot|b) \) where “\( \beta \leq b \)”, “\( \beta < b \)” and “\( b \)” are shortcuts for the events \( \{ v \in T : \beta(v) \leq b \} \), \( \{ v \in T : \beta(v) < b \} \) and \( \{ v \in T : \beta(v) = b \} \). E.g., if \( b \in S^A \) we have \( F_i(x|b) = \int \frac{\mathbf{1}_{\beta(v) = b} f(v) dv}{\int \mathbf{1}_{\beta(v) = b} f(v) dv} \). When the probability of the event we condition on is null, we will define latter properly the corresponding conditional expectations that are used for our equilibrium analysis. In few cases, some conditional expectations for which we do not provide any formal definition appear in our formulas: then any specification would work as it will be clear later since those terms are then multiplied by zero. Finally, we let \( U(v, b) \) denote the expected payoff of a buyer with type \( v \) if he bids \( b \) at the first stage (while all his opponents are using the bid function \( \beta \)) and \( S_{U}^{\max}(v) := \text{Arg max}_{b \in B_r} U(v, b) \)

\(^{12}\)Allowing for atoms in the bid distribution at some active bids burdens a bit the analysis. However, atoms at the reserve price arise commonly in auctions with externalities (Jehiel and Moldovanu [23]). We show actually that atoms at the reserve price arise very naturally here when \( N = 2 \), in particular at ex-post equilibria.
denote the set of best replies for any \( v \in T \).

We limit our analysis to equilibria that involve what we call regular bid functions: those are the bid functions \( \beta \) such that 1) \( \mathcal{P} \) is a closed set, 2) \( \beta \) is continuous on \( \mathcal{P} \), 3) \( G(.) \) has a finite number of atoms, 4) \( G(.) \) is continuously differentiable almost everywhere, 5) at any point \( b \in S^{NA} \) the function \( u \to F_i(x|\beta \leq u) \), \( i = 1, 2 \), is continuously differentiable everywhere except possibly for a finite number of points \( x \in [0, T] \). Once properly defined the corresponding derivatives are then denoted by \( \frac{\partial F_i(x|\beta \leq u)}{\partial b} \), \( i = 1, 2 \).

We strongly emphasize that we do not impose any monotonicity assumption on \( \beta \). Some weak monotonicity properties will arise endogenously from the equilibrium analysis as shown in next section.

We say that a bid function satisfies the rational participation property (henceforth the RP property) if buyers with a first valuation above [resp. strictly below] the reserve price do [resp. do not] participate in the first auction, i.e. if \( \mathcal{P} = T_r \).

**Lemma 2.1** Any equilibrium bid function \( \beta \) satisfies the rational participation property.

This implies in particular that \( \mathcal{P} \) is a convex set. Since we have assumed that \( \beta \) is continuous and that \( \mathcal{P} \) is a closed set, we obtain finally that \( S = [b, \bar{b}] \). Moreover, Lemma 2.1 implies that \( G(-1) = G(r) = F1(r) \) and thus that \( -1 \in S^A \) if and only if \( F1(r) > 0 \).

Next lemma gives some additional properties derived from the regularity assumption. For \( b, \epsilon \in \mathbb{R} \), let \( I(b, \epsilon) := \{ v \in \mathcal{P} : b \land (b+\epsilon) \leq \beta(v) \leq b \lor (b+\epsilon) \} \) and for \( v \in \mathcal{P} \) and \( \alpha > 0 \) let \( Z(v, \alpha) := \{ \tilde{v} \in \mathcal{P} : |\tilde{v}_i - v_i| \leq \alpha, i = 1, 2 \} \). Since \( \beta \) is continuous on \( \mathcal{P} \), the RP property implies then that \( Z(v, \alpha) \in \Delta(T) \).

**Lemma 2.2** For any equilibrium bid function we have: \( I(b, \epsilon) \in \Delta(T) \) for any \( b \in (b, \bar{b}) \) and \( \epsilon \neq 0 \), and then \( b \in S^{NA} \) for almost any \( b \in S \). Furthermore, we have \( \beta(v) = r \) if \( v_1 = r \) and then as a corollary \( b = r \) and \( S^* = [r, \bar{b}] \) if \( F1(r) > 0 \) [resp. \( S^* = (r, \bar{b}] \) if \( F1(r) = 0 \)].

We define below a series of objects that we should interpret as expectations conditional on the event that a buyer has bid a given amount \( b \), an event which is of measure null once \( b \) is not an atom of the CDF \( G \). Expectations conditional on zero-probability events are subject to ‘Borel’s paradox’: there are several natural ways to define conditional expectations w.r.t. to a given measure null set \( \Gamma \) as the limit of a sequence of expectations conditional on positive measure sets that converge (in some appropriate metric) to \( \Gamma \) (see Proschan and Presnell [39]). Throughout the paper and similarly to Garrett [19], all the expectations conditional on an event like \( I(b, 0) \) that we are after (in order to derive first-order conditions) will correspond to the limit when \( \epsilon \neq 0 \) goes to zero of the expectations.
conditional on the events \( I(b, \epsilon) \), a limit which is well-defined almost everywhere thanks to our regularity assumptions.\(^\text{13}\) In the body of the paper, we will often use intuitive reduced form arguments when we deal with those conditional expectations. More details on the relevance of those shortcuts are given in Appendix C.

For \( b \in S \setminus S_A \), we define the CDFs \( F_i(\cdot | b), \ i = 1, 2 \), by

\[
F_i(x | b) := \limsup_{\epsilon \to 0} F_i(x | I(b, \epsilon)). \quad (1)
\]

The function \( F_1(\cdot | b) \) [resp. \( F_2(\cdot | b) \)] corresponds to the CDF of the first [resp. second] valuation of a buyer conditional on having submitted the bid \( b \) in the first auction. If \( b \in S^{NA} \), we show in Appendix C that \( F_i(x | b) = \lim_{\epsilon \to 0} F_i(x | I(b, \epsilon)) \) and that

\[
F_i(x | b) = \frac{G(b)}{g(b)} \cdot \frac{\partial F_1(x | \beta \leq b)}{\partial b} + F_i(x | \beta \leq b) \quad (2)
\]

at any point \( x \in [0, \pi] \) except possibly on a finite number of points. From (2), we see that the role of parts 4) and 5) in the regularity assumption is to guarantee that the conditional CDF \( F_i(\cdot | b_n) \) converges weakly to \( F_i(\cdot | b) \) if the sequence \( b_n \) converges to \( b \in S^{NA} \) when \( n \) goes to infinity.

For \( b \in S^* \setminus S_A^+ \), we define the CDFs \( H_i(\cdot | b), \ i = 1, 2 \), by

\[
H_i(x | b) := [F_1(x | \beta \leq b)]^{N-2} \cdot F_i(x | b) \cdot F^{ex}(x) = \left[F_1(x | -1) \frac{G(r)}{G(b)} + \int_{b}^{r} F_1(x | u) \frac{dG(u)}{G(b)} \right]^{N-2} \cdot F_i(x | b) \cdot F^{ex}(x). \quad (3)
\]

The CDF \( H_1(\cdot | b) \) [resp. \( H_2(\cdot | b) \)] corresponds to the CDF of the highest bid in the second auction among the competitors of a given buyer and conditional on having won [resp. lost] the first auction while the highest competing bid equals \( b \) and himself having bid strictly above [resp. below] \( b \). For \( b \in S_A^+ \), such conditional CDFs are properly defined and we stick to the notation \( H_i(\cdot | b) \) to denote them. The way to derive their formal expressions, which are given in Appendix C, is more delicate due to the possibility of ties at \( b \). Similarly, for \( b \in S_A^+ \) [resp. \( b \in S_A \) and a given bidder, let \( H_1^*(\cdot | b) \) [resp. \( H_2^*(\cdot | b) \)] denote the CDF of the highest bid among his competitors in the second auction conditional on having won [resp. lost] the first auction while the highest competing bid equals \( b \) and himself having also bid

\(^{13}\)For the analysis of second-price auctions with multi-dimensional signals and in some specific models, Garrett [19] has pointed out that some previous authors made mistakes by using sequences that were not the relevant ones for their equilibrium analysis.\(^{14}\)Note that \( F_i(\cdot | b) \) is a CDF on \([0, \pi]\) since all the functions \( F_i(\cdot | I(b, \epsilon)) \) are CDFs on \([0, \pi]\). For \( b = \mathbf{b}, \mathbf{b}_i \), the corresponding definitions are \( F_i(\mathbf{x} | b) := \limsup_{\epsilon \to 0} F_i(I(b, \epsilon)) \) and \( F_i(\mathbf{x} | b) := \limsup_{\epsilon \to 0} F_i(I(\mathbf{b}, \epsilon)) \). Similarly to Lemma 2.2, we can check that \( I(b, \epsilon), I(\mathbf{b}, -\epsilon) \in \Delta(T) \) for any \( \epsilon > 0 \). When applied to \( b \in S_A^+ \), it is worthwhile to note that the definition in (1) would be consistent with the one we already gave.
In general there is a discrepancy between $H^*_1(.|b)$ and $H_i(.|b)$ and this results from a selection bias: if the highest competing bid of a given bidder equals $b \in S^A_+$, then some of the other competing bidders may also have bid $b$ with strictly positive probability and the distribution of the number of competing bidders having bid exactly $b$ depends thus on the event whether the given bidder has won for $i = 1$ [resp. lost for $i = 2$] the first auction while having bid either $b$ or a bid different than $b$. More precisely, having won [resp. lost] while being involved in a tie at the highest bid means that on average, there were less [resp. more] bidders to compete with in the tie. It is clear that this selection bias would not arise if $N = 2$: in this case we have $H_i(.|b) = H^*_i(.|b)$ for any $b \in S^A_+$.

For $b \in S^* \cup S^A_+$, we define the conditional expected payoff functions $\Pi_i(.|b)$, $i = 1, 2$, by

$$\Pi_i(x|b) := \int_0^\pi [(x - u) \vee 0] \cdot dH_i(u|b).$$

(4)

If $x \geq r$, note that we have equivalently $\Pi_i(x|b) = x \cdot H_i(x|b) - \int_0^x u \cdot dH_i(u|b)$. We should then interpret $\Pi_i(v_2|b)$ [resp. $\Pi_2(v_1|b)$] as the expected payoff from the second stage obtained by a given buyer with type $v$ conditional on the highest bid among his competitors being equal to $b$ in the first auction while having bid himself strictly above [resp. below] $b$. Similarly, let $\Pi^*_i(x|b) := \int_0^\pi [(x - u) \vee 0] \cdot dH^*_i(u|b)$ if $b \in S^A_+$ and $i = 1$ or if $b \in S^A$ and $i = 2$. Consider a bidder with type $v$, $\Pi^*_i(v_2|b)$ [resp. $\Pi^*_2(v_1|b)$] is equal to his expected payoff obtained from the second auction conditional on having won [resp. lost] the first auction while the highest bid among his competitors equals $b$ and himself having also bid $b$.

**Remark 2.2** From their interpretations, it is obvious that $H_2(.|b)$ should dominate $H_1(.|b)$ according to first-order stochastic dominance for any $b \in S^* \cup S^A_+$, which implies then that $\Pi_2(.|b) \geq \Pi_1(.|b)$. Formally, it comes from $F_2(.|E) \leq F_1(.|E)$ for any $E \in \Delta(T)$ so that $F_2(.|b) \leq F_1(.|b)$ for any $b \in S \cup S^A$.

Next lemma plays a fundamental role in our equilibrium analysis:

**Lemma 2.3** For any $b \in S^* \cup S^A_+$, the functions $x \rightarrow \Pi_1(x|b)$ and $x \rightarrow [x - \Pi_2(x|b)]$ are continuous and nondecreasing. For any $x \in [0, \pi]$ and $i = 1, 2$, the function $\Pi_i(x,.|.)$ is continuous at any point in $S^{NA}$ and is uniformly bounded.

For a given buyer having bid $b \in S^A_+$, let $p_w(b)$ denote his probability to win the first auction conditional on being involved in a tie with some of his competitors at $b$ (an event

---

15If $-1 \in S^A$, $H_2(.| - 1)$ corresponds thus to the CDF of the highest bid among the given buyer’s competitors in the second auction conditional on the fact that none of the initial buyers submit an active bid in the first auction.
which occurs with probability $G^{N-1}(b) - G^{N-1}(b)$. Conditional on such a tie, the expected payoff of this buyer if he has type $v$ equals $U_{tie}(v,b) = p_w(b) \cdot [(v_1 - b) + \Pi_1(v_2|b)] + (1 - p_w(b)) \cdot \Pi_2(v_1|b)$ where the first [resp. second] term corresponds to the case where the given buyer has won [resp. lost] the tie. Since $p_w(b)$ is a weighted sum of the scalars $\frac{1}{k}$ for $k = 2, \ldots, N$ with the respective weights corresponding to the probabilities that the tie involves $k$ bidders, we obtain then that $p_w(b) \in [\frac{1}{2}, \frac{1}{N}]$. For $N = 2$, we have thus $U_{tie}(v,b) = \frac{1}{2} \cdot [(v_1 - b) + \Pi_1(v_2|b)] + \frac{1}{2} \cdot \Pi_2(v_1|b)$. Next, we can assign any value to $U_{tie}(v,b)$ when $b \not\in S_+^A$. We now have all the ingredients for a detailed expression of the payoff function $U(v,b)$:

$$U(v,b) = [(v_1 - r) + \Pi_2^b(v_2 - 1)] \cdot G^{N-1}(r) + \int_r^{(b)} [(v_1 - u) + \Pi_1(v_2|u)] \cdot dG^{N-1}(u) +$$

$$U_{tie}(v,b) \cdot [G^{N-1}(b) - G^{N-1}(b)] + \int_{(b)}^{\bar{b}} \Pi_2(v_1|u) \cdot dG^{N-1}(u)$$

if $b \in [r, \bar{b}]$, $U(v,b) = [(v_1 - r) + \Pi_2^b(v_2 - 1)] \cdot G^{N-1}(r) + \int_r^{\bar{b}} \Pi_2(v_1|u) \cdot dG^{N-1}(u)$ if $b > \bar{b}$ and

$$U(v,-1) = \Pi_2^b(v_1 - 1) \cdot G^{N-1}(r) + \int_r^{\bar{b}} \Pi_2(v_1|u) \cdot dG^{N-1}(u).$$

**First-order conditions** The equilibrium conditions correspond to $\beta(v) \in S_{U}^{max}(v)$ for all $v \in T$. From Lemma 2.3, (5) guarantees then that $b \rightarrow U(v,b)$ is differentiable at any $b \in S^{NA}$ and $\frac{\partial U(v,b)}{\partial b} = [(v_1 - b) + \Pi_1(v_2|b) - \Pi_2(v_1|b)] \cdot (N - 1)[G(b)]^{N-2}g(b)$. For any $v \in T$, the first-order condition yields

$$b = v_1 + \Pi_1(v_2|b) - \Pi_2(v_1|b) \quad \text{if } b \in S^{NA} \cap S_{U}^{max}(v).$$

The condition holds in particular for $b = \beta(v)$ if $\beta(v) \in S^{NA}$. Consider now $b \in S_+^A \cap S_{U}^{max}(v)$ for some $v \in T$, the non-profitability of local deviations on the right leads to $U(v,b) \geq \lim_{y \rightarrow b+} U(v,\tilde{b})$ or equivalently

$$U_{tie}(v,b) \geq (v_1 - b) + \Pi_1(v_2|b) \quad \text{(8)}$$

---

16Throughout the paper, we use the notation that a bound is put in parenthesis in an integral if it is not included. This matters only when the bound corresponds to an atom of the distribution we integrate over. E.g. $\int_{a}^{b} y(u) dG(u) = \int_{a}^{b} y(u) dG(u) + y(a) \cdot (G(a) - G(a)) + y(b) \cdot (G(b) - G(b))$. According to this convention, we also consider that $\int_{a}^{b} y(u) dG(u) = \int_{a}^{b} y(u) dG(u) = 0$. 

11
while the non-profitability of local deviations on the left leads to $U(v, b) \geq \lim_{b \to -}\_ \_ \_ \_ U(v, \bar{b})$ if $b > r$ or equivalently

$$U_{tie}(v, b) \geq \Pi_2(v_1|b) \quad (9)$$

if $b > r$. We also note that the inequality

$$\Pi_2(v_1|r) \leq v_1 - r \quad (10)$$

always holds by definition for any $v \in \mathcal{T}_r$. After combining (8) and (9) if $b > r$ and (8) and (10) if $b = r$ and since we have shown in the proof of Lemma 2.1 that $S_{U}^{\text{max}}(v) = \{ -1 \}$ if $v \in \mathcal{T} \setminus \mathcal{T}_r$ or equivalently that $S_{U}^{\text{max}}(v) \cap [r, \infty) \neq \emptyset$ implies that $v \in \mathcal{T}_r,$\footnote{When we combine (8) and (10) for $r \in S_{+}^{A} \cap S_{U}^{\text{max}}(v)$ and $N = 2$, we obtain more precisely that $v_1 - r \geq \Pi_2(v_1|r) \geq v_1 - r + \Pi_1(v_2|r) \geq v_1 - r$, which also implies that $\Pi_1(v_2|r) = 0.$} we obtain that

$$U_{tie}(v, b) = \Pi_2(v_1|b) = (v_1 - b) + \Pi_1(v_2|b) \quad (11)$$

if $b \in S_{+}^{A} \cap S_{U}^{\text{max}}(v)$ and $N = 2.$

3 Characterization of the set of equilibria

Let us first introduce some additional definitions. We say that a type $v' \in \mathcal{T}$ is strongly greater than $v \in \mathcal{T}$, with the notation $v' \gg v$, if $v'_i > v_i$ for $i = 1, 2$. For $v \in \mathcal{T}$, let $L^*(v) := \{ v' \in \mathcal{T} : v \gg v' \}$ and $M^*(v) := \{ v' \in \mathcal{T} : v' \gg v \}$. We say that a type $v' \in \mathcal{T}$ is greater than $v \in \mathcal{T}$, with the notation $v' > v$, if $v'_i \geq v_i$ for $i = 1, 2$. For $v \in \mathcal{T}$, let $L(v) := \{ v' \in \mathcal{T} : v > v' \}$ and $M(v) := \{ v' \in \mathcal{T} : v' > v \}$. Note that $v' \gg v$ implies that $v' > v$ so that $L^*(v) \subseteq L(v)$ and $M^*(v) \subseteq M(v)$. For $v, v' \in \mathcal{T}$, let $\mathcal{L}(v, v') := \{ \bar{v} \in \mathcal{T} : \exists \lambda \in [0, 1] \text{ such that } \bar{v}_i = \lambda \cdot v_i + (1 - \lambda) \cdot v'_i, \text{ for } i = 1, 2 \}$ denote the straight line joining the types $v$ and $v'$. Let $\mathcal{D} := \{ v \in \mathcal{T}_r : v_1 = v_2 \} = \mathcal{L}((r, r), (\bar{v}, \bar{v}))$ denote the set of the types on the diagonal and that participate in equilibrium.

We say that a bid function satisfies the weak monotonicity property (henceforth the WM property) if for any pair $v, v' \in \mathcal{P}$, we have 1) If $v' \in L^*(v)$ [resp. $v' \in M^*(v)$], then $\beta(v) > \beta(v')$ [resp. $\beta(v) < \beta(v')$]; 2) If $x^* = \bar{\pi}$ and $v'_1 > v_1$, then $\beta(v') > \beta(v)$. See Figure 1a for an illustration.

Without any reserve price and without any extra bidder at the second stage as in Katzman [25] (i.e. if $\bar{b}_{ex} = \bar{b}^{ex} = r = 0$), the notion of efficiency is not ambiguous: it consists in assigning the two units to the two highest valuations. In this case, it is easy to check that the final allocation is always efficient if and only if the bid function satisfies the
RP and WM properties. For $N \geq 3$, the equivalence is straightforward. For $N = 2$, the equivalence comes from the fact that the final allocation is inefficient if and only if the first valuation of the winner at the first stage is strictly smaller than the second valuation of the loser. In our more general setup with possibly binding reserves and one extra bidder, the notion of allocative efficiency we consider here corresponds to the natural extension of the previous one: we proceed as if we include the valuation of the extra bidder and the reserve price (counted twice) in the pool of all valuations. We say then that the bid function is efficient if the final allocation is efficient for any realization of buyers’ types.\(^\text{18}\) It is then easy to check that efficiency implies the RP and WM properties. However, the converse does not hold for $N = 2$ since it may occur that the first valuation of the winner at the first stage lies strictly between the first and the second valuation of the loser but that the final allocation is inefficient because the second unit is actually assigned to the extra bidder.

3.1 A set of necessary conditions

In order to establish the WM property, we derive some necessary conditions on any equilibrium $\beta$ through a series of lemmas. Following a standard argument in mechanism design, we first show that if a type that is greater than another one has a best-response $b \not\in S^A$ that is strictly lower than the latter type’s best-response $b' \not\in S^A$, then both types should be indifferent between bidding $b$ or $b'$. The result extends without any restriction on $b$ and $b'$ once $N = 2$.

Lemma 3.1 Take $v, v' \in \mathcal{P}$, $b \in S_U^{\max}(v)$ and $b' \in S_U^{\max}(v')$ with $b' > b \geq r$. If $b, b' \not\in S^A$ or if $N = 2$, then $v' \in L(v)$ implies that $b' \in S_U^{\max}(v)$ and $b \in S_U^{\max}(v')$.

\(^{18}\)Once we limit ourselves to regular bid functions, the reader can check that this efficiency notion is equivalent to the weaker notion of maximization of the expected welfare ex ante. In particular, our impossibility result with risk-averse buyers in Section 4 holds also with this more common notion of efficiency.
Proof By adding the two equilibrium constraints $U(v, b) \leq U(v, b')$ and $U(v', b') \leq U(v', b)$, we obtain either if $b, b' \notin S^A$ or if $N = 2$ (by using the special expression of $U_{tie}(., .)$ when $N = 2$) that

$$\frac{1}{2} \cdot [v'_1 - \Pi_2(v'_1|b) + \Pi_1(v_2|b)] \cdot [G^{N-1}(b) - G^{N-1}(b)]$$
$$+ \int_{(b)}^{} [v'_1 - \Pi_2(v'_1|u) + \Pi_1(v_2|u)] \cdot dG^{N-1}(u)$$
$$+ \frac{1}{2} \cdot [v'_1 - \Pi_2(v'_1|b) + \Pi_1(v_2|b')] \cdot [G^{N-1}(b') - G^{N-1}(b')] \geq$$
$$\frac{1}{2} \cdot [v_1 - \Pi_2(v_1|b) + \Pi_1(v_2|b)] \cdot [G^{N-1}(b) - G^{N-1}(b)]$$
$$+ \int_{(b)}^{} [v_1 - \Pi_2(v_1|u) + \Pi_1(v_2|u)] \cdot dG^{N-1}(u)$$
$$+ \frac{1}{2} \cdot [v_1 - \Pi_2(v_1|b') + \Pi_1(v_2|b')] \cdot [G^{N-1}(b') - G^{N-1}(b')]. \tag{12}$$

If $v \succ v'$, Lemma 2.3 implies that $v'_1 - \Pi_2(v'_1|u) \leq v_1 - \Pi_2(v_1|u)$ and $\Pi_1(v_2|u) \leq \Pi_1(v_2|u)$ for any $u \in S^* \cup S_+^A$ so that the inequality (12) holds as an equality and finally that the two previous equilibrium constraints hold as equalities, or equivalently $b' \in S_{U}^{max}(v)$ and $b \in S_{U}^{max}(v')$. Q.E.D.

In the same vein, we show next that if two types have a common best-response $b \in S^{NA}$ (or $b \in S_+^A$ under the restriction $N = 2$), then it can not occur that one type is strongly greater than the other one (w.r.t. to Figure 1a, it means in particular that all the types that bid $\beta(v)$ should belong to the white area).

Lemma 3.2 Take $v, v' \in P$ and $b \in S_{U}^{max}(v) \cap S_{U}^{max}(v')$. Either if $b \in S_{NA}$ or if $N = 2$ and $b \in S^A_+$, then we have $v' \notin L^*(v) \cup M^*(v)$.

Proof We first obtain from (7) and (11) that the equality

$$v'_1 - \Pi_2(v'_1|b) + \Pi_1(v_2|b) = v_1 - \Pi_2(v_1|b) + \Pi_1(v_2|b) \tag{13}$$

holds either if $b \in S_{NA}$ or if $N = 2$ and $b \in S^A_+$. Suppose that $v \succ v'$. We have then that $v \succ v'$ which implies from Lemma 2.3 and (13) that $v'_1 - \Pi_2(v'_1|b) = v_1 - \Pi_2(v_1|b) + \Pi_1(v_2|b)$. Still from Lemma 2.3, we obtain then that the function $x \to x - \Pi_2(x|b)$ [resp. $x \to \Pi_1(x|b)$] is constant on the interval $[v'_1, v_1]$ [resp. $[v'_2, v_2]$] and thus differentiable on $(v'_1, v_1)$ [resp. $(v'_2, v_2)$]. At any point $x \geq r$ where $\Pi_1(i|b)$ (i=1,2) is differentiable, we have $\frac{d\Pi_1(x|b)}{dx} = H_i(x|b)$. Since $v_2 > v'_1$ and $v' \in P$, we obtain that $v_2 > r$ and thus that the interval $(v'_2 \lor v, v_2)$ is not empty. Finally, we obtain that $H_2(x|b) = 0$ for any $x < v_2$. Similarly, the interval $(v'_1 \lor v, v_1)$ is not empty and we have $H_2(x|b) = 1$ for any
$x > v'_1$. On the whole, this implies that $H_2(x|b) > H_1(x|b)$ for any $x \in (v'_1, v_2)$. Combined with Remark 2.2, we have raised a contradiction and we obtain thus that $v' \notin L^*(v)$. By symmetry, we have also $v \notin L^*(v')$ or equivalently $v' \notin M^*(v)$. \textbf{Q.E.D.}

As a corollary, we obtain that a buyer for which a bid $b \in S_{NA}^\ast$ (or $b \in S_{+}^A$ under the restriction $N = 2$) is a best-response should not make any profit in the second auction if he wins at the first stage while being involved in a tie with one of his opponents at $b$. On the contrary, if this buyer loses the corresponding tie at the first stage, then he is sure to have a larger valuation than the one of the winner of the tie at the second stage. Note however that it does not imply that he should win at the second stage.

\textbf{Lemma 3.3} \textit{Take $v \in P$ and $b \in S_{U}^{\max}(v)$. Either if $b \in S_{NA}$ or if $N = 2$ and $b \in S_{+}^A$, then we have: 1) $F_1(x|b) = 0$ for any $x < v_2$ and then $\Pi_1(v_2|b) = 0$; 2) $F_2(v_1|b) = 1$.}

Lemma 3.3 allows us to restate the first-order conditions as follows: for any $v \in P$ and $b \in S_{U}^{\max}(v)$ (and so in particular for the equilibrium bid $b = \beta(v)$), we have

$$b = v_1 - \Pi_2(v_1|b)$$

(14)

either if $b \in S_{NA}$ or if $N = 2$ and $b \in S_{+}^A$. If such bid $b$ is a best-response for some buyers with different first valuations, then we obtain from (14) that a buyer with a type $v$ such that $b \in S_{U}^{\max}(v)$ and who has bid $b$ and has lost the first auction with a tie is then sure to have the largest valuation at the second stage, both among the initial buyers and the extra bidder. More precisely,

\textbf{Lemma 3.4} \textit{Take $v, v' \in P$ with $v'_1 > v_1$ and $b \in S_{U}^{\max}(v) \cap S_{U}^{\max}(v')$. Either if $b \in S_{NA}$ or if $N = 2$ and $b \in S_{+}^A$, then we have $H_2(v_1|b) = 1$ and as a corollary, we obtain that $F_1(v_1|\beta < b) = 1$ if $N \geq 3$ (or equivalently $F_1(v_1|u) = 1$ for any $u \in [\epsilon, b]$) and that $F^{ex}(v_1) = 1$.}

\textbf{Proof} From (14) applied to $v$ and $v'$ and from Lemma 2.3, we obtain then that the function $x \rightarrow x - \Pi_2(x|b)$ is constant on the interval $[v_1, v'_1]$ and thus differentiable on $(v_1, v'_1)$. At any point $x \geq r$ where $\Pi_2(x|b)$ is differentiable, we have $\frac{d\Pi_2(x|b)}{dx} = H_2(x|b)$. Finally we obtain that $H_2(x|b) = 1$ for any $x \in (v_1, v'_1)$. We have thus $H_2(v_1|b) = 1$. From (3) for $b \in S_{NA}$ and from (32) for $N = 2$ and $b \in S_{+}^A$, we obtain then that $F^{ex}(v_1) = 1$, $F_2(v_1|b) = 1$ (which was already known from Lemma 3.3) and if $N \geq 3$, $F_1(v_1|\beta < b) = F_1(v_1|1) \cdot \left(\frac{G(-1)}{\epsilon(b)}\right) + f_r(b) F_1(v_1|u) \frac{d\tilde{G}(u)}{\epsilon(b)} = 1$. This last equality implies that $F_1(v_1|I(u, \epsilon)) = 1$ for any $r \leq u < b$ and $\epsilon > 0$ such that $u + \epsilon < b$. From (1), we obtain then that $F_1(v_1|u) = 1$ for any $u \in [\epsilon, b]$. \textbf{Q.E.D.}
Lemma 3.4 is then used as the main argument to show next that buyers with the same first valuation have the same equilibrium bid under certain conditions, a property which is one of the main steps to obtain equilibrium uniqueness.

**Lemma 3.5** Take \( v, v' \in \mathcal{P} \) with \( v_1 = v'_1 \). If either \( F_{\text{ex}}(v_1) < 1 \) or \( N \geq 3 \), then \( \beta(v) = \beta(v') \).

Lemma 3.5 implies that the a priori ad hoc restriction made by Katzman [25] (that the bid function \( \beta \) is based solely on the first valuation) is not restrictive if \( x^* = \overline{v} \). In other words, under this condition, any equilibrium candidate is fully characterized by the function \( \hat{\beta} : [r, \overline{v}] \rightarrow \mathcal{S} \) where \( \hat{\beta}(x) := \beta(x, x) \).

Next lemma, which is illustrated in Figure 1b, states that if two types have the same equilibrium bid \( b \) then any type lying between those two types according to the order \( \gg \) should also bid \( b \).

**Lemma 3.6** Take \( \tilde{v}, \tilde{v}' \in \mathcal{P} \) with \( \tilde{v}' \gg \tilde{v} \). If \( \beta(\tilde{v}) = \beta(\tilde{v}') = b \), then \( \beta(v) = b \) for any \( v \in L^*(\tilde{v}') \cap M^*(\tilde{v}) \) and as a corollary \( b \in \mathcal{S}^A \) and \( N \geq 3 \).

If we apply Lemma 3.6 to pairs \( \tilde{v}, \tilde{v}' \in \mathcal{D} \), we obtain that \( \hat{\beta} \) is strictly increasing once \( N = 2 \). We show more generally that \( \hat{\beta} \) is a bijection from \( [r, \overline{v}] \) to \( [r, \overline{b}] \) for any \( N \). Let \( \hat{\beta}^{-1} : [r, \overline{b}] \rightarrow [r, \overline{v}] \) denote the corresponding inverse function.

**Lemma 3.7** \( \hat{\beta} \) is strictly increasing and \( \overline{b} = \hat{\beta}(\overline{v}) > r = \hat{\beta}(r) \).

If we combine Lemma 3.7 with Lemma 3.5, we obtain the WM property when \( x^* = \overline{v} \). In this case, we have furthermore \( G(b) = F1(\hat{\beta}^{-1}(b)) \) for any \( b \in [r, \overline{b}] \) and the bid distribution has no atom in \( [r, \overline{b}] \). When \( x^* < \overline{v} \) (which guarantees that \( N = 2 \)), we obtain from Lemma 3.6 that for any \( v, v' \in \mathcal{P} \), we have \( \beta(v) \neq \beta(v') \) if \( v \gg v' \). If there exist \( v, v' \in \mathcal{P} \) such that \( v > v' \) and \( \beta(v) < \beta(v') \), then by continuity (recall that \( \hat{\beta}(r) = r \)) there exists \( \tilde{v} \in L((r, r), v') \subseteq L^*(v) \) such that \( \beta(\tilde{v}) = \beta(v) \) which would raise a contradiction as argued above. On the whole we have shown that \( \beta(v) > \beta(v') \) once \( v \gg v' \). To summarize,

**Proposition 3.1** Any equilibrium bid function satisfies the weak monotonicity property.

The RP and WM properties for a given bid function \( \beta \) implies per se (i.e. independently of \( \beta \) being an equilibrium or not) that: 1) If a type \( v \) buyer bid \( \beta(v) \) or more and wins at the first stage while paying \( b \geq \beta(v) \), then at the second stage he will not strictly outbid the initial buyer that has bid \( b \) at the first stage. 2) If a type \( v \) buyer bid according to \( \beta(v) \) or less and loses at the first stage while the highest bid was below \( \beta(v) \), then he has necessary
the highest current valuation among the initial buyers. In the same vein as Lemma 3.3, this further implies a bunch of useful properties for the continuation values as summarized in next lemma.

**Lemma 3.8** For any bid function \( \beta \) satisfying the RP and WM properties, we have: 1) \( \hat{\beta}(v_2 \vee r) \leq \beta(v_2 \vee r) \leq \hat{\beta}(v_1) \), for any \( v \in T_r \); 2) \( \hat{\beta} \) is strictly increasing and if \( x^* = \tau \), then \( S^A \cap U = \emptyset \); 3) \( H_1(x|b) = 0 \) for any \( b \in S^* \cap S^A \) and \( x < \hat{\beta}^{-1}(b) \vee \underline{b}^* \) and then \( \Pi_1(v_2|b) = 0 \) if \( v_2 \leq \hat{\beta}^{-1}(b) \vee \underline{b}^* \); 4) \( H_2(x|b) = F^{ex}(x) \) for any \( b \in S^* \cup S^A \) and \( x \geq \hat{\beta}^{-1}(b) \) and then \( \Pi_2(v_1|b) = v_1 F^{ex}(v_1) - \int_0^{v_1} u dH_2(u|b) \) if \( v_1 \geq \hat{\beta}^{-1}(b) \). If \( v_1 \geq \hat{\beta}^{-1}(b) > \bar{b}^* \), then

\[
\Pi_2(v_1|b) - \Pi_2(\hat{\beta}^{-1}(b)|b) = v_1 - \hat{\beta}^{-1}(b). \tag{15}
\]

**Remark 3.1** From Proposition 3.1 and part 2) of Lemma 3.8, we obtain that \( S^A \cap U = \emptyset \) if \( N \geq 3 \) and thus that (11) holds for any \( b \in S^A \cap S^{max}_U \). From (5), discontinuities in the payoff functions \( U(v,.) \) can arise only at atoms of the bid function and thus only when \( x^* < \tau \). From (11), \( U(v,.) \) is continuous at any bid \( b \in S^{max}_U \). On the whole discontinuities can occur only when \( x^* < \tau \) and once \( b \in S^A \setminus S^{max}_U \).

From Lemma 3.8, the first-order condition (14) can be restated as follows: if \( \beta(v) \in S^{NA} \cup S^A \), we have

\[
\beta(v) = v_1(1 - F^{ex}(v_1)) + \int_0^{v_1} u dH_2(u|\beta(v)). \tag{16}
\]

This last property is now called the local martingale property (henceforth the LM property). This terminology comes from the following interpretation: if two buyers were in tie at a bid \( b \in S^{NA} \cup S^A \), then the expected price at the second auction equals \( b \). Eq. (16) is actually a bit stronger insofar as it states that for any realization of the type \( v \) of the loser of the tie then the expected price of the second auction equals \( \beta(v) \). Since this buyer has necessary the highest current valuation among the initial buyers, then in the second auction he could be outbid only by the extra bidder, in which event the price would equal \( v_1 \). On the whole, we should interpret the right-hand side in (16) as follows: with probability \( (1 - F^{ex}(v_1)) \) [resp. \( F^{ex}(v_1) \)], the extra bidder [resp. the loser of the tie at the first stage] submits the highest bid in the second auction and the price equals \( v_1 \) [resp. the expected price equals \( \int_0^{v_1} u \frac{dH_2(u|\beta(v))}{\frac{dH_2(u|\beta(v))}{\int_0^{v_1} |dH_2(u|\beta(v))|}} \).]

The LM property implies that \( \beta(v) \leq v_1 \) for almost all \( v \in T_r \) and \( \beta(v) = v_1 \) for almost all \( v \in T_r \cap L((\underline{b}^{ex}, \bar{b}^{ex})) \). Since \( \beta \) is continuous, this is true for all \( v \) satisfying the previous conditions.
Corollary 3.1 For any \( v \in \mathcal{T}_r \) [resp. \( v \in \mathcal{T}_r \cap L((\hat{b}^e, \hat{b}^e)) \)], we have \( \beta(v) \leq v_1 \) [resp. \( \beta(v) = v_1 \)].

Next lemma states that the property that buyers’ bid function depends solely on their first valuation, and is strictly increasing in it, carries over to environment with \( N = 2 \) for those buyers that run the risk to be outbid by the extra bidder at the second stage. This lemma is a key ingredient to establish allocative efficiency. We show a bit more in next lemma by giving the explicit necessary form of the equilibrium bid in those cases.

Lemma 3.9 The bid distribution \( G \) contains no atoms on \([r, \hat{\beta}(x^*)]) \cup \{\hat{b}\} \). On the set \( \mathcal{T}_r \cap L((x^*, x^*)) \), the bidding function \( \beta \) is strictly increasing in \( v_1 \), does not depend on \( v_2 \) so that \( \beta(v) = \hat{\beta}(v_1) \) and is given by\(^{19}\)

\[
\beta(v) = v_1 - \int_r^{v_1} \left( \frac{F_1(u)}{F_1(v_1)} \right)^{N-2} F_2(u|v_1)F^e(u)du. \tag{17}
\]

For any \( v \in \mathcal{T}_r \cap L((x^*, x^*)) \), we have \( \beta(v) \geq \hat{\beta}(x^*) \).

Remark 3.2 From the proof of Lemma 3.9, we also get that (17) is satisfied for any \( v \in \mathcal{T}_r \cap L((x, x)) \) in any equilibrium bid function such that \( \beta(v) = \hat{\beta}(v_1) \) for any \( v \in L((x, x)) \).

3.2 Equilibrium properties

With the help of the previous lemmas and before establishing the existence of an equilibrium and characterizing the set of equilibria, we are able to provide a series of interesting properties satisfied by all equilibria. First, the final allocation is always efficient. Second, the sequence of prices is a submartingale under a mild additional assumption. Third, we characterize the sets of buyers’ best replies which take the form of intervals. Those sets do not reduce to singletons when there are no extra bidders at the second stage. Katzman \[25\] has already emphasized those three features in the two special equilibria he exhibits.

Proposition 3.2 [Efficiency] Equilibrium bid functions are efficient. As a corollary, all equilibria raise the same expected revenue as the Vickrey auction.\(^{20,21}\)

\(^{19}\)The reader can check that (17) corresponds to a generalization of formula (1) in Katzman \[25\].

\(^{20}\)We consider that the Vickrey auction is defined according to the assumption that the reserve price \( r \) corresponds to the seller’s reservation value for each item. See also Ausubel \[1\] for a dynamic implementation of the Vickrey auction that suits to our model.

\(^{21}\)If there are possibly several extra bidders entering the second auction (what we actually exclude to simplify the exposition), it would not be possible to assign the two units to those extra bidders and then efficiency could fail since those latter bidders may have the two highest valuations. Nevertheless, some “constrained” efficiency would still prevail.
Proof When \( x^* = \bar{r} \), efficiency is a corollary of the RP and WM properties: 1) the first unit is assigned to the agent with the highest first valuation (which corresponds to the seller in the case where all buyers have a first valuation strictly below \( r \)); 2) at the second stage, the second unit is assigned to the agent with the highest remaining valuation and we are thus done. On the contrary, the argument is slightly more subtle in the case \( N = 2 \) and \( b^{ex} < \bar{r} \) since the first unit may not be assigned to the agent with the highest first valuation. In the event where the first unit is assigned to the agent with highest first valuation, then efficiency is guaranteed as above. Consider now the remaining events (given the RP property): suppose that there exists \( v, v' \in \mathcal{T}_r \) such that \( \beta(v) \geq \beta(v')(\geq r) \) and \( v_1 < v'_1 \). From Lemma 3.9, we must have \( v_1 \geq b^{ex} \), i.e. the winner’s first valuation is always above the extra bidder’s valuation. On the whole to obtain efficiency it is sufficient to show that \( v'_2 \leq v_1 \) which comes from part 1) of the WM property.

Revenue equivalence results then from interim payoff equivalence which holds for any mechanism that yields the same allocation of the items and leaves no rent to the buyers with types in \( L((r, r)) \) (see Krishna and Perry [26] for a payoff/revenue equivalence theorem with multi-dimensional signals, note in particular that we use that \( \mathcal{T}_r \) is convex in order to fit into their framework). Q.E.D.

Remark 3.3 In presence of allocative inefficiencies and if resale were allowed, then the equilibrium analysis would depend on the details of the aftermarket rules. Furthermore, ‘ratchet effects’ associated to resale opportunities typically preclude the existence of pure separating equilibria in auctions, and the way the ratchet effect works depends crucially on the disclosure rules about the submitted bids (see Lebrun [30] for first-price auctions). In two-stage sequential second-price auctions, if the bid functions at the first stage were opening the door to allocative inefficiencies, then bidding up to the current valuation at the second stage would not have any equilibrium status. On the contrary, the efficiency property guarantees that the equilibria of the game without resale, and in particular our assumption that buyers bid their valuation at the second stage, are robust to environments with resale opportunities independently of the way resale is modeled.

Remark 3.4 If \( N = 2 \) and \( r = 0 \), we can check that for any realization of buyers’ types then the seller’s revenue is higher in the Vickrey auction when the two units are bundled (i.e. a single second-price auction for the bundle and without any reserve price) than in the Vickrey auction when the two units are sold separately (or equivalently the Ausubel’s [1] auction for the two units). From Proposition 3.2, we obtain then that the seller’s expected revenue in the two-stage sequential second-price auction is lower than in the second-price
auction for the bundle in this case. Intuitively, this revenue comparison reverses itself as $N$ gets larger.\footnote{Palfrey \cite{Palfrey} and Chakraborty \cite{Chakraborty} establish similar results when buyers have additives preferences.}

In our setup with possibly binding reserve prices, some units may remain unsold. If an item is not sold, we adopt the convention that its price is equal to the reserve price. Let $v^{(1:N)}$ denote the type of the winning bidder if the first unit is sold (otherwise we let $v^{(1:N)} := (r, r)$). Let $v^{(2:N)}$ denote the type of the second-highest bidder if there were at least two active bidders in the first stage (otherwise we let $v^{(2:N)} := (r, r)$). Let $b^e$ denote the realization of the bid of the extra bidder. For $i = 1, 2$, let $p^i$ denote the price at the $i^{th}$ stage. According to our convention, we have thus $p^1 = \beta(v^{(2:N)}) \geq r$ and $p^2 \geq r$. The price $p^1$ is thus distributed according to the CDF $G^*(u)$ where $G^*(u) = 0$ if $u < r$ and $G^*(u) = NG^{N-1}(u) - (N - 1)G^N(u)$ otherwise. With probability one, we have $p^1 \in S^{NA} \cup S^{A+} \cup \{r\}$. Furthermore, if the same buyer wins in both stages, we have also $p^2 = v^{(2:N)}_1 \lor b^e$. From Corollary 3.1, we obtain then that $p^2 \geq p^1$ in those events. This gives a first channel how multi-unit demand is at the source of the departure from Milgrom-Weber’s \cite{MilgromWeber} martingale property which holds in the symmetric independent private value paradigm under single-unit demand.

**Proposition 3.3 [Price trends]** Suppose that $F_2(.|b)$ dominates $F_2(.|b')$ according to first-order stochastic dominance for any $b, b' \in S^{NA} \cup S^{A+}$ with $b > b'$, then prices follow a submartingale:

$$E[p^2|p^1] \geq p^1$$

for any $p^1 \in S^{NA} \cup S^{A+} \cup \{r\}$ and so with probability one. As a corollary, prices are increasing on average: $E[p^2] \geq E[p^1]$.

The additional assumption on $F_2(.|b)$ in the proposition is intuitive: it says that we expect buyers submitting larger bids to have larger second valuations. However, it is not related to the primitive of the model. We will see later how to apply this proposition for a specific equilibrium, which corresponds actually to the equilibrium once we establish uniqueness.

**Remark 3.5** It is obvious that the submartingale property would hold a fortiori if we restrict ourselves to the subsample of auctions where both units have been sold.

The submartingale property is somehow puzzling if we have in mind the highly publicized declining price anomaly found in many empirical studies. This deserves several
comments. On the one hand, many empirical studies are devoted to heterogenous goods that are possibly ordered by value which yields naturally decreasing price patterns (see e.g. Beggs and Graddy [2]) while studies with homogenous units involve also sequences with more than two units and examine price trends while mixing various sequences with different number of units (see e.g. van den Berg et al. [40]). In a nutshell, our submartingale property for sequential auctions with multi-unit demands holds for a game that does not necessary fit with the underlying game of many empirical data analyzed so far. Furthermore, instead of testing formally a submartingale property like $E[p_2|p_1] \geq p_1$ or $E[p^2] \geq E[p^1]$, those works are testing rather whether $E[p^2] \geq 1$, $E[\log(p^2)] \geq 0$ or $E[1[p^2 \geq p^1]] \geq \frac{1}{2}$ as in the seminal analysis of McAfee and Vincent [32]. On the other hand, Remark 2.1 gives us some clues why we should be cautious about the way to interpret martingale properties with respect to raw data. According to those perspectives, what we refer as the price at the second stage in the present analysis, i.e. $p_2$, does not correspond to the price usually reported by analysts. E.g., we have $p_2 = v_2(1:N) \vee v_1^{(2:N)}$ in the case where only one unit is put for sale while analysts would leave the price at the second stage as a missing value. If what we call the extra bidder results from stochastic supply and if we restrict ourselves to the price sequences where two units have been put for sale, then there are good chances to observe rather declining prices. This discussion carries over if what we call the extra bidder results from differences in term of quality between the two units.

The static Vickrey auction has the desirable feature that efficiency is obtained in an equilibrium where bidders are using weakly dominant strategies. There is a literature proposing generalizations of the English auction when multiple goods are for sale (e.g. Ausubel [1]). The aim is to replicate the outcome of the Vickrey auction with an ascending auction format, exactly as the English auction does w.r.t. the second-price auction. Nevertheless, some ascending Vickrey auctions suffer from indifferences: at some stage in the ascending path, bidders are indifferent between a large set of bids (see Lamy [29] for more details and references on this issue). The two-stage sequential auction does not necessary implement the Vickrey payoffs. However, since it induces the same (efficient) assignment, it is equivalent to the Vickrey auction from an interim perspective. This is also a mechanism where buyers express their preferences in a dynamic way and this can be perceived as beneficial in the same way as ascending Vickrey auctions are often perceived as improvements w.r.t.

---

23 Another departure from our model of two-stage sequential auctions and that is often observed in auction houses as Christie’s and Sotheby’s is the so-called “buyer’s option” where the winner of the first stage has the option to purchase the second unit at the first auction’s price. This rule plays in favor of declining prices sequence as argued by Black and de Meza [5].

24 We guess that the main reason for this is to increase the power of the tests, i.e. the ability to reject the null hypothesis, since those studies typically involve small samples.

25 In this vein, Jeitschko [24] shows a supermartingale property under single-unit demand.
their static counterpart. However, we see below that it may suffer from indifferences depending on the active (or not) role of perturbation by extra bidders. In particular, if $\bar{b}^{ex} = r$ as in Katzman [25], then a type $v$ buyer is indifferent between all possible bids in the interval $[\hat{\beta}(v_2), \hat{\beta}(v_1)]$. See Figure 2 for an illustration.

**Proposition 3.4** [Characterization of the sets of buyers’ best replies] For any $v \in T$, we have

$$S_{U}^{max}(v) = \begin{cases} [\hat{\beta}(v_2 \lor (\bar{b}^{ex} \land v_1)), \hat{\beta}(v_1)] & \text{if } v_2 \lor \bar{b}^{ex} > r \\ \{-1\} \cup [r, \hat{\beta}(v_1)] & \text{if } v_2 \leq r = \bar{b}^{ex} \end{cases}$$

(18)

if $v_1 \in (r, \bar{v})$ and $S_{U}^{max}(v) = \{-1\}$ if $v_1 < r$. In particular, $S_{U}^{max}(v)$ is a singleton if $v_1 \in (r, \bar{b}^{ex}]$ and $S_{U}^{max}(v)$ is not a singleton if $v_1 > \bar{b}^{ex} \lor v_2$. \(^{27}\)

Figure 2: Expected payoff from choice of bid (if $v_2 > \bar{b}^{ex}$)

---

26 See Lamy [29] for some references and an overview on the benefits of ascending formats.

27 To be exhaustive about the characterization of $S_{U}^{max}(v)$ for any $v \in T$, we have $S_{U}^{max}(v) = \{-1, r\}$ if $v_1 = r$, $S_{U}^{max}(v) = [\hat{\beta}(v_2 \lor (\bar{b}^{ex} \land v_1)), \infty)$ if $v_1 = \bar{v}$ and $v_2 \lor \bar{b}^{ex} > r$, and $S_{U}^{max}(v) = B_r$ if $v_1 = \bar{v}$ and $v_2 \leq r = \bar{b}^{ex}$.
a winner who has bid $b \in [\hat{\beta}(v_1), \hat{\beta}(v_1) + \epsilon]$, the given buyer with first valuation $v_1$ should lose the second auction with some positive probability, i.e. $\overline{B}(v) \leq \hat{\beta}(v_1) + \epsilon$. Since this is true for any $\epsilon > 0$, we obtain that $\overline{B}(v) \leq \hat{\beta}(v_1)$. With a similar argument, we can show that $\overline{B}(v) = \hat{\beta}(v_2 \vee (\overline{b}^{ex} \wedge v_1))$.

3.3 Equilibrium characterization, existence, uniqueness and multiplicity

3.3.1 Characterization

We have established in Subsection 3.1 that any equilibrium should necessarily satisfy the RP, WM and LM properties. Those properties are actually sufficient and thus characterize the set of equilibria.

**Proposition 3.5** The regular bid function $\beta$ is an equilibrium if and only if it satisfies the rational participation, the weak monotonicity and the local martingale properties.

Let us now give an alternative characterization stated in terms of “isobid curves” and which is slightly easier to check and to interpret graphically. For any bid function $\beta$ and $x \in [r, \overline{v}]$, we let $I_\beta(x) := I(\hat{\beta}(x), 0)$ the set of types that bid as the type $(x, x)$. For any equilibrium, the RP property and lemma 3.7 guarantee that the isobid curves $I_\beta(x)$, $x \in [r, \overline{v}]$, form a partition of $T_r$.

**Proposition 3.6** [Characterization in terms of isobid curves] The regular bid function $\beta$ is an equilibrium if and only if it satisfies the rational participation property, $\hat{\beta}$ is strictly increasing,

$$
I_\beta(x) = \{v \in T : v_1 = x\} \quad \text{if } N \geq 3 \quad \text{or if } N = 2 \quad \text{and } x < \overline{b}^{ex}
$$

$$
I_\beta(x) \subseteq T_r \setminus \{L^*(((x, x)) \cup M^*((x, x)))\} \quad \text{if } N = 2 \quad \text{and } x \geq \overline{b}^{ex}
$$

(19)

for any $x \in [r, \overline{v}]$, $\bigcup_{x \in [r, \overline{v}]} I_\beta(x) = T_r$ and

$$
\hat{\beta}(x) = x(1 - F^{ex}(x)) + \int_0^x udH_2(u|\hat{\beta}(x))
$$

(20)

for any $x \in [r, \overline{v}]$ such that $\hat{\beta}(x) \in S^{NA} \cup S^A_+$.

In a nutshell, this characterization says that a bid function is an equilibrium if and only if it efficient while eq. (16) from the LM property is satisfied for the types on $D$. Apart from saying that it is sufficient to check that the first-order condition holds for types on $D$, this last characterization gives also a better intuition of the source of equilibrium uniqueness/multiplicity. For $N = 2$ and $x \geq \overline{b}^{ex}$, we see from (19) that we have a lot
of freedom in the way to specify the isobid curve \( I_\beta(x) \) while maintaining efficiency. In Subsection 3.3.3, we will see how to construct a continuum of equilibria parameterized by a real number: the corresponding isobid curves are straight lines. Next we use the terminology linear bid functions or linear equilibria. See Figure 3 for an illustration. However, it is also clear from (19) that we have much more degree of freedom so that linear bid functions are a very limited class of equilibrium candidates once \( x^* < \varpi \).

**Remark 3.6** Pick \( \tilde{r} \in [r, b^{\text{ex}}] \) with \( \tilde{r} < \varpi \). Consider then the new environment such that: the reserve price in the two auctions equals \( \tilde{r} \), buyers’ valuations are distributed according to the CDF \( \tilde{F} := \frac{F(u) - F(\tilde{r})}{\tilde{F}(\tilde{r})} \lor 0 \) whose support is denoted by \( \tilde{T} = T_{\tilde{r}} \), the CDF of the extra bidder’s valuation is given by \( \tilde{F}^{\text{ex}}(b) := F^{\text{ex}}(b) \lor 1[b \geq \tilde{r}] \). We have thus \( \tilde{F}_i(.|E) = F_i(.|E) \) for any \( E \subseteq T_{\tilde{r}} \) with \( E \in \Delta(T) \). We can easily check that if the original environment fits in our model then the new environment also does. Let \( \tilde{\beta} : \tilde{T} \rightarrow B_{\tilde{r}} \) denote an equilibrium candidate. For any equilibrium \( \beta : T \rightarrow B_r \) in the original environment, we have \( \beta(v) \geq \tilde{r} \) if and only if \( v \in T_{\tilde{r}} \). If \( \tilde{\beta} \) and \( \beta \) coincide on \( \tilde{T} \), then \( H_2(.|b) \) and \( \tilde{H}_2(.|b) \) coincide for any \( b \in (\tilde{r}, \tilde{b}) \) (for any \( b \in [\tilde{r}, \tilde{b}] \) if \( \tilde{r} \in S^A \)) which implies then that for any \( b \in (\tilde{r}, \tilde{b}) \) (for any \( b \in [\tilde{r}, \tilde{b}] \) if \( \tilde{r} \in S^A \), if (16) holds for \( \beta \) in the original environment then it also holds for \( \tilde{\beta} \) in the new environment and conversely. From Proposition 3.6, we obtain on the whole that \( \tilde{\beta} \) is an equilibrium (in the new environment) if and only if \( \tilde{\beta}(v) = \beta(v) \) for any \( v \in T_{\tilde{r}} \), where \( \beta \) is an equilibrium (in the original environment). Consider then such an equilibrium \( \tilde{\beta} \) and a given buyer with type \( v \in T \setminus T_{\tilde{r}} \), his unique best-response among the bids in \( [\tilde{r}, \infty) \) is then \( \tilde{r} \) (see the proof of Proposition 3.4). Consider now the slightly modified game (in the new environment) where bidders are constrained to participate, i.e. to submit a bid in the interval \( [\tilde{r}, \infty) \), and let us consider a (regular) bid function \( \tilde{\beta} : T \rightarrow [\tilde{r}, \infty) \). From our previous discussion, we have that \( \tilde{\beta} \) is an equilibrium if and only if \( \tilde{\beta}(.) = \beta(.) \lor \tilde{r} \) where \( \beta \) is an equilibrium in the original environment. This result is useful for our analysis of English auctions.

### 3.3.2 Existence and uniqueness

Katzman [25] establishes the existence of an equilibrium where the bid function is based solely on the first valuation. See Figure 3b. From Remark 3.2, any equilibrium where the bid function is based solely on the first valuation is given by \( \beta(v) = -1 \) if \( v_1 < r \) and \( \beta(v) = \psi(v_1) \) otherwise where \( \psi(x) := x - \int_r^x \left[ \frac{F_1(u)}{F_1(x)} \right]^{N-2} F_2(u|x) F^{\text{ex}}(u)du \) for any \( x \in [r, \varpi] \). Note that \( \psi \) is continuous and \( \psi(r) = r \). Let \( \psi^{-1} : [r, \psi(\varpi)] \rightarrow [r, \varpi] \) denote the inverse function of \( \psi \) once \( \psi \) is strictly increasing. In our more general framework, this equilibrium candidate is actually a (regular) equilibrium once we introduce a mild
additional assumption.

**Assumption A 1** The function $\psi$ is continuously differentiable on $(r, v]$ with $\psi' > 0$.

Next lemma provides a simple and easily interpretable condition guaranteeing that A1 holds: conditional on having a higher first valuation, buyers should have higher second valuations.

**Assumption A 2** The CDF $F_2(:, x)$ dominates $F_2(:, x')$ according to first-order stochastic dominance for any $x, x' \in [r, v]$ with $x \geq x'$, i.e. $F_2(u|x) \leq F_2(u|x')$ for any $u$.

In Katzman [25], the sampling scheme satisfies the constraint $F_2(u|x) = \left[ F_1(x') \right]^{1/2} \cdot F_2(u|x')$ (if $x, x' > 0$) so that A2 holds and thus also A1 from next lemma.

**Lemma 3.10** Assumption A2 implies A1.

If $N = 2$, Katzman [25] establishes the existence of another equilibrium where the bid function is based solely on second valuations. Liu [31] shows that this equilibrium is an ex-post equilibrium and that for any realization of buyers’ types, the final payoffs coincide with the ones of the Vickrey auction. We generalize this ex-post equilibrium when there are possibly binding reserve prices and once $b^{ex} = b^{ex} = r'$, which can also be viewed as the case where the reserve price at the second stage $r'$ is larger than the one at the first stage. See Figure 3c for an illustration when $r' = r$.

**Proposition 3.7** [Existence]

- Under A1, an equilibrium exists in which the bid function, denoted by $\beta_R1$, is based solely on the first valuation and where

$$
\beta_R1(v) = v_1 - \int_r^{v_1} \left[ \frac{F_1(u)}{F_1(v_1)} \right]^{N-2} F_2(u|v_1) F^{ex}(u) du \tag{21}
$$

if $v \in T_r$ and $\beta_R1(v) = -1$ otherwise. This equilibrium is called the **R1-equilibrium**.

- If $N = 2$ and $b^{ex} = b^{ex} = r'$, an equilibrium exists in which the bid function, denoted by $\beta_R2$, is based solely on the second [resp. the first] valuation when the first valuation is above [resp. below] $r'$ and where

More generally, Liu [31] constructs explicitly an equilibrium satisfying those properties for sequential second-price auctions with two bidders having nonincreasing marginal valuations and for any number of identical items for sale and with no reserve price. In the two units case, the equilibrium constructed by Liu [31] corresponds precisely to the equilibrium based on second valuations exhibited by Katzman [25].
\[
\beta_{R2}(v) = v_2 \lor (v' \land v_1)
\] 

(22)

if \(v \in \mathcal{T}_r\) and \(\beta_{R2}(v) = -1\) otherwise. This is an ex-post equilibrium and the final payoffs coincide with the ones of the Vickrey auction. This equilibrium is called the R2-equilibrium.\(^{29}\)

From Lemma 3.9 we already know that any equilibrium bid function \(\beta\) should satisfy \(\beta(v) = \beta_{R1}(v)\) if \(v \in L((x^*, x^*))\). Beyond R2-equilibria and once \(x^* < \tau\), we can seek more generally for equilibria where the bid function is based solely on the second valuation on the set \(\mathcal{T}_r \setminus L((\bar{b}^{ex}, \bar{b}^{ex}))\). By continuity, we obtain then that \(\beta(v) = \tilde{\beta}_{R1}(\bar{b}^{ex})\) if \(v \in \mathcal{T}_r \setminus \{L^*((\bar{b}^{ex}, \bar{b}^{ex})) \cup M^*((\bar{b}^{ex}, \bar{b}^{ex}))\}\). Since \(\tilde{\beta}\) is strictly increasing, we have also that \(H_2(u|\beta(v)) = 1[u \geq v_2]\) for any \(u\) and \(v \in \mathcal{T}\) such that \(v_2 > \bar{b}^{ex}\). From (16), we obtain finally that \(\beta(v) = v_2\) if \(v_2 > \bar{b}^{ex}\). By continuity, we must also have \(\tilde{\beta}(\bar{b}^{ex}) = \bar{b}^{ex}\). Furthermore, from (21) we have that \(\tilde{\beta}_{R1}(\bar{b}^{ex}) = \bar{b}^{ex}\) if and only if \(\bar{b}^{ex} = \bar{b}^{ex}\). We obtain finally that the kind of equilibria we were looking for exist only when \(\bar{b}^{ex} = \bar{b}^{ex}\) and that they correspond to the R2-equilibrium in this case.

**Remark 3.7** If \(\bar{b}^{ex} < \tau\), we get equivalently from the previous discussion that equilibrium bid functions such that \(\beta(v) = \tilde{\beta}\left(v_2 \lor (\bar{b}^{ex} \land v_1)\right)\) for any \(v \in \mathcal{T}_r\) do not exist if \(\bar{b}^{ex} < x^*\) and coincide with the R2-equilibrium otherwise.

**Remark 3.8** Since our model assumes that \(f > 0\) on \(\mathcal{T}_r\), it does not encompass either the flat demand case where \(F2(u|v_1) = 1[u \geq v_1]\) for any \(v_1 \in [0, \tau]\) or the single-unit demand

\(^{29}\)If \(r' = \tau\), then the R1- and R2-equilibria coincide (we have \(\beta_{R1}(v) = \beta_{R2}(v) = v_1\)): the setup corresponds then to one with a single unit for sale.
case where \( F_2(0|v_1) = 1 \) for any \( v_1 \in [0, v] \). In the former case, we can see that (21) reduces to \( \beta_{R1}(v) = v_1 \) which corresponds to the unique equilibrium. In this later case, we can see that (21) reduces to \( \beta_{R1}(v) = v_1 - \int_r^{v_1} \left( \frac{F_1(u)}{F_1(v_1)} \right)^{N-2} F_{ex}(u) du \), an expression which corresponds actually to the unique symmetric equilibrium. We can also check that Proposition 3.4 also holds with single-unit demands. Note that in the special case where \( b_{ex} = b_{ex} = r = 0 \) then the previous expression is equivalent to the formula derived by McAfee and Vincent [32] with risk-neutral buyers and buyers are then indifferent between all bids below their equilibrium bid. Those indifferences are actually the key ingredient that explains why the equilibrium fails to be robust against small perturbations from the risk-neutral setup. In Section 4, we delineate how the various equilibria of our models with multi-unit demands may fail to be robust with respect to risk aversion.

Katzman [25] has shown that the R1- and R2- equilibria are the unique equilibria with so-called separable bid functions, i.e. such that active bids depend solely on the first or the second valuation. From the remarks 3.2 and 3.7, we obtain a similar result in our generalized setup. However, if \( x^* = v \), we obtain a much more general uniqueness result: from Lemma 3.9, the bid function \( \beta_{R1} \) is the unique equilibrium candidate among all regular bid functions. Furthermore, if the function \( \psi \) is not strictly increasing, then \( \beta_{R1} \) does not satisfy the WM property so that it can not be an equilibrium. To summarize,

**Proposition 3.8 (Uniqueness)** If \( x^* = v \), then the \( R1 \)-equilibrium is the unique equilibrium if \( A1 \) holds, while there is no equilibrium if the function \( \psi \) fails to be strictly increasing.

This result is extremely important for empirical research because it yields identification. Lamy [28] establishes that the CDF \( F \) is nonparametrically identified from the winning price at both stages if bidders are assumed to play the R1-equilibrium. As a corollary of Proposition 3.8, we obtain thus that the model is identified if \( x^* = v \) and so once \( N \geq 3 \).

---

\(^{30}\)Lamy [27] has shed some light on the indifference property under single-unit demand and how it translates in term of non-robustness of the equilibrium. Contrary to the present setup with multi-unit demands and where buyers are always assumed to be symmetric, Lamy [27] considers also an extension with asymmetric buyers: the indifference property carries over for at least one buyer (more precisely the buyer which is sure to win at the second stage in case of a tie) which is then sufficient to make the equilibrium non-robust to risk-aversion.

\(^{31}\)From Lemma 3.9, we have similarly that any equilibrium bid function \( \beta \) should coincide with \( \beta_{R1} \) for first valuations below \( b^{ex} \). This implies also that there are no equilibria if \( \psi \) fails to be strictly increasing on \([r, b^{ex}]\).

\(^{32}\)More precisely, the analysis in Lamy [28] is limited to the cases where \( b^{ex} = r = 0 \). However, the identification result extends for binding reserve prices once we care only on the identification of \( F \) on \( T_r \). Similarly, the identification result extends also to setup with extra bidders (we emphasize that we assume here that bidders’ identities are known to the analyst so that she can distinguish extra bidders from initial bidders).
To end this subsection, let us detail a bit how propositions 3.3 and 3.4 work with the R1- and R2- equilibria.

**Proposition 3.9** In the R1-equilibrium, the sequence of prices is a submartingale under A2. In the R2-equilibrium, the sequence of prices is nondecreasing for any realization of buyers’ types and thus a fortiori a submartingale.

In his model and for the R1-equilibrium, Katzman [25] has pointed out that the price sequence is a martingale conditional on the event where different buyers win at stages one and two. This property does not extend to our framework. More details on price trends are given in Appendix K.1 and in particular how the above martingale property noted by Katzman [25] turns to a submartingale property under our more general sampling schemes.

For a given equilibrium bid function $\beta$, we say that buyers bid at the top [resp. buyers bid at the bottom] if $\beta(v) = \max\{b \in S^\text{max}_v\}$ for almost all $v \in \mathcal{P}$ [resp. $\beta(v) = \min\{b \in S^\text{max}_v\} \setminus \{-1\}$] for almost all $v \in \mathcal{P}$. In other words, saying that buyers bid at the top [resp. the bottom] means that active bidders submit the largest [resp. smallest] active bid among their set of best replies. It is straightforward from Proposition 3.4 that buyers bid at the top [resp. the bottom] in the R1-equilibrium [resp. the R2-equilibrium]. Next proposition says more by establishing converses and thus provides a status w.r.t. indifferences of those two kinds of equilibria.

**Proposition 3.10** In a given equilibrium $\beta$: 1) buyers bid at the top if and only if $\beta = \beta_{R1}$; 2) buyers bid at the bottom if and only if either $b^{ex} = v$ and $\beta = \beta_{R1}$ or $N = 2$, $b^{ex} < v$ and $\beta = \beta_{R2}$.

### 3.3.3 Equilibrium multiplicity

Throughout this subsection, we assume that $N = 2$ and $b^{ex} = r$. If A1 holds, then we face equilibrium multiplicity since we are under conditions where both the R1- and the R2-equilibrium exist as already pointed by Katzman [25]. We will show that the equilibrium multiplicity caveat is even more severe since we face a continuum of equilibria. We consider below a special class of equilibrium candidates where the isobid curves are linear functions and are parameterized by $\alpha \in [0, 1]$. See Figure 3a for an illustration. For a given $\alpha \in [0, 1]$ and $x \geq r$, we define $\xi_\alpha(x)$ as follows

$$
\xi_\alpha(x) := \begin{cases} 
\{v \in \mathcal{T} : \alpha v_1 + (1 - \alpha)v_2 \leq r\} & \text{if } x = r \\
\{(1 - (1 - \alpha)(1 - t)), (1 - t)x) : t \in [0, 1]\} & \text{if } r < x < \alpha v \\
\{(x + t(v - x), x - \frac{\alpha}{1 - \alpha}t(v - x)) : t \in [0, 1]\} & \text{if } x \in [\alpha v, v] \text{ and } x > r 
\end{cases}
$$

(23)
We can easily check that \( \bigcup_{x \in [r, v]} \xi_\alpha(x) = T_r \) and that \( \xi_\alpha(x) \subseteq T_r \setminus \{ L^*(x, x) \cup M^*((x, x)) \} \) for any \( x \in [r, v] \).

For a given \( \alpha \in [0, 1] \), we let \( \beta_\alpha \) denote the bid function that is characterized by the following properties: it satisfies the RP property and \( I_{\beta_\alpha}(x) = \xi_\alpha(x) \) and \( \hat{\beta}_\alpha(x) = \int_0^x (u \lor r) d F_2(u \mid \hat{\beta}_\alpha(x)) \) for any \( x \in [r, v] \). Note that we have \( \beta_0 = \beta_{R2} \) and \( \beta_1 = \beta_{R1} \). From Proposition 3.6, the bid function \( \beta_\alpha \) is an equilibrium if and only if it is regular and \( \hat{\beta}_\alpha \) is strictly increasing on \([r, v]\). Concerning regularity, it is obtained from the monotonicity of \( \hat{\beta}_\alpha \) in the same way as in the proof of Proposition 3.7. Before our general result, let us illustrate our construction with a special case: if \( f \) is the uniform density on \( T_0 \) and for any \( r < v \), then the equilibrium candidates \( \beta_\alpha \) are actually equilibria for any \( \alpha \in [0, 1] \). This is verified in Appendix S where explicit formulas are given for this special case.

We show below that a continuum of equilibria prevails more generally. The proof consist in showing that the bid function \( \beta_\alpha \) is an equilibrium bid function once \( \alpha \) is close enough to zero. In other words, the class of equilibria we construct are perturbations of the R2-equilibrium.

**Proposition 3.11** [Continuum of equilibria] If \( N = 2 \) and \( b^{ex} = r > 0 \) and if we also assume that the density \( f \) is continuously differentiable on \( T_r \), then there is a continuum of equilibria.

### 3.4 Equilibrium selection

If we have in mind that the number of potential buyers is strictly larger than two in almost all applications, then equilibrium multiplicity in second-price auctions should not be an important issue for econometricians and as noted earlier, the model is thus nonparametrically identified in most environments. However, since the multiplicity issue associated to the case \( N = 2 \) will reemerge for any number of bidders under sequential English auctions (with observable dropout prices) as argued in Section 5, we explore a bit equilibrium selection criteria that reduce the set of equilibria to a singleton when \( N = 2 \).

**Robustness to perturbations of the game**  Contrary to Katzman [25], our analysis covers also models with possibly some extra bidders entering the game at the second stage. This feature can also be used as a refinement tool. To refine the set of equilibria we can limit ourselves to equilibria whose bid function corresponds to the limit of equilibrium bid functions of perturbated models characterized the CDFs \( (F_j^{ex})_{j \in N} \) such that the sequence

---

33Remark 3.4 provides a theoretical argument why two-stage sequential auctions should not be very popular when \( N = 2 \). E.g., we have \( N = 7 \) in the fish auctions analyzed by Brendstrup [6] and Brendstrup and Paarsch [7].
\( \mathcal{F}_j^{\text{ex}} \) converges weakly to \( F^{\text{ex}} \) as \( j \) goes to infinity. If we use the sequence \( \mathcal{F}_j^{\text{ex}}(x) := \alpha_j \cdot F^{\text{ex}}(x) + (1 - \alpha_j) \cdot 1[x \geq \overline{v}] \) with \( \lim_{j \to \infty} \alpha_j = 1 \) and \( \alpha_j \in (0, 1) \) for any \( j \in \mathbb{N} \), then we obtain from Proposition 3.8 that only the R1-equilibrium can be suitable. Furthermore, from (21) and from the portmanteau theorem, the R1-equilibrium corresponds to the limit of the perturbated models. We point out that perturbations with extra bidders can receive alternative interpretations (as noted earlier in Remark 2.1).\(^{34}\) In this vein, arbitrarily small differences in term of quality between the two units allow us to select the R1-equilibrium.

Up to now, we have also considered that the number of potential buyers is fixed and common knowledge ex ante. Let us briefly discuss a model with a stochastic number of potential buyers as in McAfee and McMillan [33]. Let \( \lambda_k \in [0, 1] \) denote the probability to face \( k \in \mathbb{N} \) potential competitors and assume that \( \lambda_k > 0 \) for at least one \( k \geq 2 \). The generalization of the first-order conditions (7) and (11) leads to:\(^{35}\)

\[
b = v_1 + \sum_{k=1}^{\infty} \frac{\lambda_k}{1 - \lambda_0} \cdot \Pi_1^k(v_2|b) - \sum_{k=1}^{\infty} \frac{\lambda_k}{1 - \lambda_0} \cdot \Pi_2^k(v_1|b) \tag{24}
\]

for any \( v \in T_r \) either if \( b \in S_{U}^{NA} \cap S_{U}^{\max}(v) \) or if \( N = 2 \) and \( b \in S_{U}^{A} \cap S_{U}^{\max}(v) \). After noting that Lemma 2.3 holds if we replace the payoff function \( \Pi_i \) by \( \tilde{\Pi}_i \) \( (i = 1, 2) \), we obtain that the lemmas 3.1-3.4 extend straightforwardly to the environment with a stochastic number of buyers. In particular, we have \( \tilde{\Pi}_1(v_2|b) = 0 \) for any \( b \in S_{U}^{NA} \cap S_{U}^{\max}(v) \).

The next key step to see why our analysis carries over is that \( \frac{d\Pi_1}{dx}(x|b) = 1 \) is equivalent to \( \sum_{k=1}^{\infty} \frac{\lambda_k}{1 - \lambda_0} \cdot [F_1(x|\beta < b)]^{k-1} \cdot F_2(x|b) \cdot F^{\text{ex}}(x) = 1 \) which implies then that \( F_1(x|\beta < b) = 1 \) since there exists \( k \geq 2 \) such that \( \lambda_k > 0 \). As in Lemma 3.5, this yields that any equilibrium bid function should depend solely on the first valuation. The first-order condition (24) leads then to the following generalization of the R1-equilibrium:

\[
\beta(v) = v_1 - \sum_{k=1}^{\infty} \frac{\lambda_k}{1 - \lambda_0} \cdot \int_{r}^{v_1} \left[ \frac{F_1(u)}{F_1(v_1)} \right]^{k-1} F_2(u|v_1) F^{\text{ex}}(u) du \tag{25}
\]

for \( v \in T_r \) and \( \beta(v) = -1 \) otherwise. We can check that this unique equilibrium candidate is actually an equilibrium if and only if the right-hand side of (25) is strictly increasing as a function of \( v_1 \) on the range \([r, \overline{v}]\). On the whole, uncertainty w.r.t to the set of potential participants selects the R1-equilibrium.

\(^{34}\)Those interpretations may be much more convincing in our refinement perspective, in particular because rigid registration policies may preclude new bidders to arrive at the second stage without having being registered at the first stage.

\(^{35}\)We now put a superscript indicating the (realized) number of opponents \( k \) for the expected payoff functions \( \Pi_1 \) and \( \Pi_2 \).
Robustness to perturbations of the beliefs A popular notion in the literature is the ‘ex-post equilibrium’ concept requiring that the equilibrium strategies in the game of incomplete of information remain an equilibrium in the game where buyers’ types were common knowledge. As pointed out in Proposition 3.7, R2-equilibria are ex-post equilibria. Furthermore, we show below that those are the unique equilibria that are ex-post equilibria.

On the one hand, in any equilibrium we must have $\beta(v) = v_1$ if $v_1 \in [r, b_\text{ex}]$ and $\beta(v) \geq v_1$ if $v_1 > b_\text{ex}$ (Lemma 3.9). On the other hand, let us show that in any ex-post equilibrium we must have $\beta(v) = v_2 \lor b_\text{ex}$ if $v_1 > b_\text{ex}$. Consider an equilibrium bid $b > b_\text{ex}$ so that there exists $v, v' \in T \setminus L((b_\text{ex}, b_\text{ex}))$ such that $\beta(v) = \beta(v') = b$ which implies from the WM property that $v_1 \geq v'_2$ and $v'_1 \geq v_2$. Consider a realization of buyers’ valuations such that two buyers $1, 2$ have respectively the valuations $v$ and $v'$ so that they both bid $b$ in equilibrium, while the remaining buyers (including the extra bidder) have all the valuation $(r, r)$. If $b > v'_2$ [resp. $b < v'_2$], buyer 1 would strictly benefit by bidding slightly less [resp. more] than $b$. On the whole we obtain that $b = v'_2$, and by symmetry also $b = v_2$ and thus that $v_2 = v'_2$. We obtain finally that the set $\{x \in [0, \overbar{v}] : \exists u \in [b_\text{ex} \lor x, \overbar{v}] \text{ with } \beta(u, x) = b\}$ is a singleton for all bids $b \in (b_\text{ex}, \overbar{v})$. Since $\hat{\beta}$ is strictly increasing while $\beta$ is continuous, we have then $\beta(v) = \hat{\beta}(v_2) \lor b_\text{ex}$ for any $v \in T \setminus L((b_\text{ex}, b_\text{ex}))$. From Lemma 3.9, we have also $\beta(v) = \hat{\beta}(v_1) = \beta_{R_1}(v)$ if $v_1 \leq x^*$ so that we have necessary $b_\text{ex} = x^*$. Then we obtain that $\beta(v) = \hat{\beta}(v_2 \lor (b_\text{ex} \land v_1))$ for any $v \in T_r$. From Remark 3.7, we conclude that any ex-post equilibrium should coincide with an R2-equilibrium.

It is remarkable that the two previous families of refinements select those two equilibria that are somehow two extremities in the set of equilibria. In light of Proposition 3.10 and in the cases where we face equilibrium multiplicity, the R1-equilibrium [resp. R2-equilibrium] if it exists can also be viewed as the unique equilibrium such that buyers bid at the top [resp. the bottom]. Nevertheless, the theoretical status of such a selection criterion that assumes that bidders are somehow aggressive [resp. cautious] insofar as they prefer to bid respectively more [resp. less] in cases of indifferences, is unclear. However, in the perspective of a dynamic version of the auction (second-price auctions can also be viewed as an open auction where dropout prices are unobservable), intuition suggests that the “bid at the top” refinement can emerge naturally as the result of an option value to wait as in Compte and Jehiel [13].

---

36In Compte and Jehiel’s [13] English auction model, bidders may refine their valuations over time through exogenous private information so that they bid strictly above their expected value in equilibrium.
4 Risk aversion

As ex-post equilibria, R2-equilibria remain equilibria if buyers are risk-averse. For any given equilibrium bid function $\beta$ with risk-neutral buyers and for a given specification of risk-aversion, we can wonder whether there exists an equilibrium bid function $\beta^*$ so that its isobid curves coincide with the ones of $\beta$. In other words, we can wonder whether there is a strictly increasing function $\phi: \mathcal{B}_r \rightarrow \mathcal{B}_r$ such that $\beta^*(v) = \phi(\beta(v))$ for all $v \in \mathcal{T}$. When $\overline{b}^e < \overline{v}$ and for utility functions displaying decreasing absolute risk aversion, we will see that the answer is always no except for the R2-equilibrium in which case we must have $b_{ex} = b_{ex}$ and $N = 2$ while $\phi$ is the identity function. In the perspective of Proposition 3.10, this means also that any set of isobid curves which result from an equilibrium under risk neutrality and where buyers do not bid at the bottom can not sustain an equilibrium with natural specifications of risk-aversion. We show actually a slightly stronger result: we establish the non-existence of efficient equilibria under fairly general assumptions. The analysis follows closely the arguments developed by McAfee and Vincent [32] who establish the non-existence of an efficient equilibrium in two-stage sequential second-price auctions with single-unit demands and from which we borrow some technical lemmas. We bring also something new by shedding some light on the fundamental cause of non-robustness to risk-aversion and how it can be alleviated by the presence of extra bidders. Nevertheless, an exhaustive equilibrium analysis under risk-aversion is beyond the scope of this work.

We consider next that buyers are risk-averse with a von Neuman-Morgensten utility function $V(.)$ satisfying the following assumption: $V(.)$ is three times continuously differentiable and satisfies $V' > 0, V'' < 0$ and $V(0) = 0$. A utility function $V$ displays decreasing [resp. nondecreasing, constant] absolute risk aversion if $-\frac{V''}{V'}$ is decreasing [resp. nondecreasing, constant]. Then we say briefly that $V$ is DARA [resp. NDARA, CARA]. For any (regular) bid function $\beta$, we have

$$\frac{\partial U(v, b)}{\partial b} = \left[ \int_0^v V(v_1 - b + ([v_2 - u] \vee 0))dH_1(u|b) - \int_0^v V([v_1 - u] \vee 0)dH_2(u|b) \right] \cdot (N - 1)G^{N - 2}(b)g(b)$$

$$\frac{\partial^2 U(v, b)}{\partial v_1 \partial b} = \left[ \int_0^v V'(v_1 - b + ([v_2 - u] \vee 0))dH_1(u|b) - \int_0^{v_1} V'(v_1 - u \vee 0)dH_2(u|b) \right] \cdot (N - 1)G^{N - 2}(b)g(b)$$

for any $v \in \mathcal{T}$ and $b \in S^{NA}$. If we assume furthermore that $\beta$ is efficient then it satisfies the RP and WM properties and we obtain finally from Lemma 3.8 that $H_1(u|\beta(v)) = 0$ if $u < v_2$ and $H_2(u|\beta(v)) = F_{ex}(u)$ if $u \geq v_1$, once $\beta(v) \in S^* \cup S^+_A$. For any $v \in \mathcal{T}_r$ such that $\beta(v) \in S^{NA}$, the first-order condition $(\frac{\partial U(v, \beta(v))}{\partial b} = 0)$ and the second-order condition $(\frac{\partial^2 U(v, \beta(v))}{\partial v_1 \partial b} \geq 0)$ that any candidate to be an efficient equilibrium has to satisfy are then
respectively

\[
\mathcal{V}(v_1 - \beta(v)) = \int_{0}^{\bar{v}} \mathcal{V}([v_1 - u] \vee 0)dH_2(u|\beta(v)), \tag{27a}
\]

\[
\mathcal{V}'(v_1 - \beta(v)) \geq \int_{0}^{\bar{v}} \mathcal{V}'([v_1 - u] \vee 0)dH_2(u|\beta(v)) - \mathcal{V}'(0)(1 - F^{\text{ex}}(v_1)). \tag{27b}
\]

The key technical argument (see the Appendix in McAfee and Vincent [32]) is the following: if \( \mathcal{V} \) is DARA, then \( \mathcal{V}(c) = E[\mathcal{V}(Y)] \) implies \( \mathcal{V}'(c) < E[\mathcal{V}'(Y)] \) for all random variable \( Y \).\footnote{We say that a stochastic variable is a random variable if its CDF is not a unit step function or equivalently if it is not a deterministic variable.}

Consider first an efficient equilibrium bid function such that the stochastic variable \([v_1 - u] \vee 0\) where \( u \) is distributed according to \( H_2(.,|\beta(v)) \) is a random variable for a given \( v \in T \) with \( \beta(v) \in S^{NA} \) and \( v_1 \geq \bar{b}^{ex} \). We obtain then that for any DARA utility function, if the first-order condition (27a) holds then the second-order condition (27b) should fail. Note that this argument precludes directly the existence of an equilibrium where the bid function depends solely on the first valuation once \( \bar{b}^{ex} < \bar{v} \). In particular, this precludes the existence of an efficient equilibrium if \( N \geq 3 \). The proof for \( N = 2 \) is more subtle and is thus relegated to Appendix T.

**Proposition 4.1** [Non-existence of efficient equilibria] Suppose that \( \mathcal{V} \) is DARA and that \( \bar{b}^{ex} < \bar{v} \). Then the only efficient equilibrium bid functions are the one corresponding to R2-equilibria. As a corollary, if either \( \bar{b}^{ex} < \bar{b}^{ex} \) or \( N \geq 3 \), then any equilibrium suffers from allocative inefficiencies.

Proposition 4.1 is thus an extension to environments with multi-unit demands of the non-existence result of efficient equilibria derived by McAfee and Vincent [32] under single-unit demand. As in McAfee and Vincent [32], we have on the contrary an existence result if \( \mathcal{V} \) is NDARA. A generalization of the R1-equilibrium with risk-neutral buyers is a suitable solution.

**Proposition 4.2** Suppose that \( \mathcal{V} \) is NDARA and that A2 holds. Then an efficient equilibrium exists in which the bid function, denoted by \( \beta^{\mathcal{V}}_{R1} \), is based solely on the first valuation and where

\[
\beta^{\mathcal{V}}_{R1}(v) = v_1 - \mathcal{V}^{-1}\left(\int_{r}^{v_1} \mathcal{V}(v_1 - u)d\left[\frac{F_1(u)}{F_1(v_1)}\right]^{N-2}F_2(u|v_1)F^{\text{ex}}(u)\right) \tag{28}
\]

for \( v \in T_r \) and \( \beta^{\mathcal{V}}_{R1}(v_1) = -1 \) otherwise. If \( x^* = \bar{v} \), then it is the unique efficient equilibrium.
NDARA utility functions is a form of risk-aversion which is not perceived as the most relevant one in the literature. It includes however as a limit case the popular class of CARA utility functions.

In order to check that the bid function \( \beta_{VR} \) is an (efficient) equilibrium, the proof relies on establishing inequalities which have all the form

\[
1 - \frac{E[V'(Y)]}{V(E[Y])} + (1 - \lambda F^e(x)) \cdot \frac{V'(0)}{V(E[Y])} \geq 0
\]

where \( \lambda \in [0, 1], \ x < v \) and \( Y \) is a stochastic variable such that \( V'(E[Y]) = E[V'(Y)] \), the second-order condition (27b) being a special case. The second term in (29) is always positive while the first term is guaranteed to be positive only when \( V \) is NDARA. If \( \eta := \lim_{x \to v} F^e(x) < 1 \), then the presence of extra bidders have a stabilizing effect insofar as it makes the second term being strictly positive: if \( V \) is sufficiently close to a CARA utility function, then the bid function \( \beta_{VR} \) is an equilibrium. More formally, we can check that for any stochastic variable \( Y \), we have \( V'(E[Y]) \geq E[V'(Y)] - 2\omega \cdot V'(0) \), where \( \omega := \sup_{x \in \mathbb{R}} |V(x) - \left[ 1 - \frac{V'(x)}{V'(0)} \right]|. \) Note that \( \omega = 0 \) if and only if \( V \) is CARA. Then \( \omega \leq \frac{(1-\eta)}{2} \) is a sufficient condition for \( \beta_{VR} \) being an equilibrium since it guarantees that the second term in (29) compensates the first term when this latter is negative.

## 5 English auctions

Formally, second-price auctions are almost never used in real-life. The practical relevance of this format results from the fact that it is strategically equivalent to some of the various models of the open ascending price auction.\(^{38}\) Consider more precisely ascending ‘button’ auction models: the auctioneer raises the price continuously, starting at price \( r = 0 \).\(^{39}\) A bidder indicates whether he is interested in buying the object at the current price by (de)pressing a button. Exits are assumed to be irrevocable. The auction ends when only one bidder remains. Various specifications can be done to complete the definition of the auction. Such variations depend in particular on the information available to the bidders. Here we consider the two most popular models: first, the model where bidders’ actions are fully private, i.e. dropout prices are unobservable, so that the only information that any bidder knows when deciding whether to stay active or not at a given price is that there is at least one other active bidder in the auction; second, the model where dropout prices are

\(^{38}\)See Bikhchandani and Riley [4] for a discussion on the way to model open auctions.

\(^{39}\)To simplify we consider no reserve price in this section. We have thus also \( T = T_0 \). The equilibria derived in Proposition 5.1 can be adapted easily to incorporate a positive reserve price.
observable. With only two buyers, both models are obviously equivalent. More generally, the first model is strategically equivalent to the second-price auction while the second model corresponds to what is usually called the English auction in the literature. We stick to this terminology. Under pure private values, the second-price and the English auction lead to the same unique equilibrium in undominated strategies where buyers bid their valuations. The picture is very different under interdependent values. In a pure common value model with symmetric buyers and when there are at least three buyers, Bikhchandani and Riley [4] show that there is a unique equilibrium in the second-price auction while a continuum of (possibly asymmetric) equilibria prevails in the English auction. When there are only two buyers, equilibrium multiplicity prevails (in both models) in their framework as in ours. The key point is then that in an English auction, the subgame after all buyers except two drop out is strategically equivalent to a second-price auction with two buyers (with modified initial conditions and by interpreting nonparticipation as immediate dropouts), a notable point which holds very generally and thus also in our sequential auction framework so that our previous analysis with two buyers can be applied. Bikhchandani and Riley [4] obtain then that for any number of buyers, the English auction is contaminated by the equilibrium multiplicity phenomenon that arises with only two buyers. The same phenomenon arises also here. Note however, that while the various equilibria in Bikhchandani and Riley [4] may lead to different assignments of the good, the final assignment of the two units is the same in all the equilibria derived in Proposition 5.1. More precisely, those equilibria always yield an efficient assignment and are thus interim payoff equivalent to the Vickrey auction.

More precisely, the English auction is conducted according to the following rules. The price begins at zero with all bidders “in.” Bidders may then exit observably and irreversibly as the price rises continuously and exogenously. The auction ends when only one bidder remains, with this bidder receiving the good and paying the price at which his final opponent quit. If at some point in the auction all remaining bidders quit simultaneously, one of these bidders is selected at random and named the winner at his exit price. (Bikhchandani et al. [3])

The auction can be viewed in a sequence of phases, beginning in phase zero and entering a new phase each time a bidder exits. In phase $1 \leq k \leq N - 2$, let $B^k = (b_1, \ldots, b_k)$ denote the vector of the exit prices at which bidders quit the auction. Let $\mathcal{B}^k := \{B^k \in \mathbb{R}^k : 0 \leq b_1 \leq \ldots \leq b_k < \bar{v}\}$ and $\overline{\mathcal{B}}^k := \{B^k \in \mathbb{R}^k : 0 \leq b_1 \leq \ldots \leq b_k\}$. According to the auction rule, we have always $B^k \in \overline{\mathcal{B}}^k$. Let $B^0 := \emptyset$ and $b_0 = 0$. A buyer’s (pure) strategy for phase $0 \leq k \leq N - 2$ and for the vector of exit prices $B^k \in \overline{\mathcal{B}}^k$ is given by a bid function

\[\text{Since we are in a symmetric environment, it does not matter whether bidders’ identities are observed or not.}\]
\( \beta^k(\cdot; B^k) : T \to [b_k, \infty) \) specifying his exit price as a function of his type.

Characterizing the set of symmetric equilibria for English auctions lies beyond the scope of this work. Proposition 5.1 derives a large class of regular symmetric equilibria.\(^{41}\)

**Proposition 5.1** Assume that \( N \geq 3 \). The strategies \( \beta = (\beta^0, \beta^1, \ldots, \beta^{N-2}) \) where

(i) \( \beta^k(v; B^k) = b_k \vee v \) for any \( v \in T \), \( B^k \in \mathfrak{B}^k \) and \( k = 0, \ldots, N - 3 \);

(ii) If \( B^{N-2} \in \mathfrak{B}^{N-2} \), then \( \beta^{N-2}(\cdot; B^{N-2}) = \bar{\beta}(\cdot) \vee b_{N-2} \), where \( \bar{\beta}(\cdot) \) is an equilibrium bid function of the second-price auction with two buyers with types distributed according to \( F \), with no reserve price and with the CDF of the valuation of the extra bidder in the second auction being equal to \( F^{ex}(b) \cdot 1[b \geq b_{N-2}] \). If \( B^{N-2} \in \mathfrak{B}^{N-2} \setminus \mathfrak{B}^{N-2} \), then \( \beta^{N-2}(\cdot; B^{N-2}) = b_{N-2} \).

constitute a perfect Bayesian equilibrium (PBE).

From condition (i) and from Bayesian updating, the beliefs of the two remaining bidders are the following on the equilibrium path once we have reached the final phase (phase \( N-2 \)) and when \( b_{N-2} < \bar{v} \): it is common knowledge that the highest first valuation among the \( N - 2 \) other initial buyers that exit the auction is equal to \( b_{N-2} \) while the type of the remaining buyer to bid against is drawn from the CDF \( \frac{F(\cdot) - F(b_{N-2}, b_{N-2})}{1 - F(b_{N-2}, b_{N-2})} \). Given that \( (\beta^0, \beta^1, \ldots, \beta^{N-3}) \) satisfy the condition (i), we obtain then from Remark 3.6 that \( \beta \) constitutes an equilibrium in the subgame at phase \( N-2 \) if and only if condition (ii) holds.

In the same way as Bikhchandani et al. [3] have shown (in a slightly different model) that there is a lot of degree of freedom in the way to specify the equilibrium strategy of the bidders at the phase \( k = 0, \ldots, N - 3 \) in English auctions, condition (i) can be weakened considerably. More precisely, Proposition 5.1 can be generalized if we replace condition (i) by condition (i’) for \( \beta^0, \ldots, \beta^{N-3} \) saying (informally) that: exit prices reveal buyers’ first valuations, buyers with lower first valuations drop out earlier and buyers never bid strictly above their first valuations.\(^{42}\) As in Bikhchandani et al. [3], this means that one must be cautious in how to interpret inframarginal bids. It is left to the reader to check that this larger set of equilibria has actually a theoretical status: it corresponds to the set of efficient (regular symmetric) equilibria. This is straightforward after having checked that efficiency implies condition (i’).

**Remark 5.1** If \( \bar{b}^{ex} = b^{ex} \), then a suitable solution for the equilibrium \( \bar{\beta}(\cdot) \) in condition (ii) is the R2-equilibrium. In this manner and in the same way as in Bikhchandani et al. [3],

\(^{41}\)By regularity, we mean here that \( \beta^k(\cdot; B^k) \) is a regular bid function for any \( k \) and any \( B^k \in \mathfrak{B}^k \).

\(^{42}\)A formal proof would require a mild extension of Remark 3.6.
the resulting strategy $\beta$ is also an “ex-post equilibrium” and we emphasize that this holds for any number of buyers.43

6 Conclusion

From an empirical perspective, our equilibrium uniqueness result opens the door to the identification of buyers’ valuation distribution as argued previously. Furthermore, this allows us to run counterfactuals once buyers’ valuation distribution has been estimated: e.g. we can compute the seller’s expected revenue under alternative reserve price policies or if the two units are bundled in a single package, i.e. kinds of exercises that were not contemplated by previous empirical works on sequential auctions. Nevertheless, some equilibrium properties prompt us to remain cautious (even when the equilibrium is unique): we have established in Proposition 3.4 that buyers are typically indifferent between a large set of bids, which suggests that it could be difficult to learn how to play an equilibrium even for experienced bidders. Whether the equilibrium behavior assumption is a reasonable one or not is a clear subject for further research, in particular through lab experiments.

References


43In extensive games, it would be natural to combine the ex-post equilibrium concept with sequential rationality as in Börgers and McQuade [8]. In the same way as they discuss the equilibria in Bikhchandani and Riley [4] and Bikhchandani et al. [3], we conjecture that the aforementioned ex-post equilibrium is a *weakly information-invariant sequential equilibrium* according to their terminology.


Appendix

A  Proof of Lemma 2.1

As preliminaries, let us make few remarks on the function $U(v, b)$.

(a) Consider $v \in T$ such that $v_1 \leq r$. The continuation payoff from the second stage is always null. Then we obtain that $U(v, -1) = 0$ and that $U(v, b) \leq \int_r (v_1 - u) dG_{N-1}(u) \leq (v_1 - r) \cdot G_{N-1}(r) \leq 0$ for any $b \geq r$. The last inequality can be interpreted as follows: a buyer bidding $b$ will win the auction with a probability which is greater than $G_{N-1}(r)$ and will lose at least $r - v_1$ in this case (in any other case where he wins at the first stage, then his payoff is negative).

(b) Consider $v \in T$ such that $v_1 \geq r$. When we compare the expected payoffs $U(v, -1)$ and $U(v, r)$, there are two relevant events for the realization of the highest opponent bid that makes a difference: either it is equal to $-1$ or it is equal to $r$ and the realization of those events enter in the payoff difference only if the corresponding bid is in $S^A$. Conditional on either the realization $r$ or the realization $-1$, the expected payoff from winning at the first stage is at least $v_1 - r$ which is greater than any payoff that can be obtained from the second stage when losing. If $-1 \in S^A$, let $\Pi^*_2(v_1 | -1)$ denote the expected payoff from the second stage of a bidder that did not participate at the first stage conditional on the event that none of his opponents has participated in the first stage (this is consistent with some notation we introduce later). If $-1 \notin S^A$, let $\Pi^*_2(v_1 | -1)$ be any value. We obtain then that

$$U(v, r) - U(v, -1) \geq [(v_1 - r) - \Pi^*_2(v_1 | -1)] \cdot G_{N-1}(r). \quad (30)$$

Suppose that there exists $v \in T \setminus T_r$ and $b \in S^{max}_U(v)$ with $b \geq r$. It implies in particular that $r > 0$. From the preliminary remark (a), we obtain that $U(v, b) = 0$ and thus that $G(r) = 0$. In other words, since the set $\mathcal{P}$ is assumed to be closed, all bidders must participate to the first auction. Since $v \in T \setminus T_r$, there is a neighborhood of $v$ that is a subset of $T \setminus T_r$ and that belongs to $\Delta(T)$. On the whole, the event where the types of all buyers belong to this set will occur with positive probability which implies that some bidders with $v_1 < r$ should win the first auction with some positive probability. However, in such a case they would pay at least $r$ and make negative profits, which would raise a contradiction. We have thus shown that $v \in T \setminus T_r$ implies that $S^{max}_U(v) = \{-1\}$ so that $\beta(v) = -1$, or equivalently $\mathcal{P} \subseteq T_r$.  

40
Suppose that \( T_r \setminus P \in \Delta(T) \) which guarantees that \( G(r) > 0 \) (and so that \(-1 \in S^A\)) and that there exists \( v \in T \) such that \( v_1 > r \) and \( \beta(v) = -1 \). For any \( x > r \) (and thus in particular for \( x = v_1 \)), this implies also that \( \Pi_2(x) < x - r \) since if the given buyer wins the second unit, then he would have to pay strictly more than \( r \) with positive probability. Combining the previous inequalities with (30) for \( v \), we obtain that \( U(v, r) - U(v, 1) > 0 \) which raises a contradiction with \(-1 \in S_{U}^{\max}(v)\). On the whole we have shown that there is no positive mass of types in \( T_r \) that do not participate. Thus any type in \( T_r \) can be viewed as a limit of a sequence of types that belong to \( P \). Since \( P \) is closed, we have thus shown that \( v_1 \geq r \Rightarrow \beta(v) \geq r \), or equivalently \( T_r \subseteq P \).

**B Proof of Lemma 2.2**

Consider \( b \in (\underline{b}, \overline{b}) \) and \( \epsilon \neq 0 \). Take \( b^* \in (b \land (b + \epsilon), b \lor (b + \epsilon)) \cap (\underline{b}, \overline{b}) \neq \emptyset \) and \( v^* \in P \) such that \( \beta(v^*) = b^* \). Since \( v^* \in P = T_r \), then \( Z(v^*, \alpha) \in \Delta(T) \) for any \( \alpha > 0 \). Furthermore, since \( \beta \) is continuous on \( P \), there exists \( \alpha > 0 \) such that \( Z(v^*, \alpha) \subseteq I(b, \epsilon) \). We obtain finally that \( \text{Prob}(I(b, \epsilon)) \geq \int \int 1[Z(v^*, \alpha)]f(v)dv > 0 \). We obtain as a corollary that \( G(\cdot) \) is strictly increasing on \([\underline{b}, \overline{b}]\) and thus that \( G \) is differentiable with \( g(b) > 0 \) for almost any \( b \in [\underline{b}, \overline{b}] \) and finally that almost any \( b \in S \) belongs to \( S^{NA} \).

Suppose now that there exists \( v \) such that \( v_1 = r \) and \( \beta(v) > r \). From the preliminary remark (a) in the proof of Lemma 2.1, we obtain that \( 0 = U(v, \beta(v)) \leq \int_r^{\beta(v)} (r - u)dG^{N-1}(u) \leq 0 \) and thus that \( G(\beta(v)) = G(r) \). Since we have shown that \( G(\cdot) \) is strictly increasing on \([\underline{b}, \overline{b}]\), then we obtain that \( \beta(v) = \overline{b} > r \). Let \( \overline{x} := \frac{\overline{b} + r}{2} \in (r, \overline{b}) \). Since \( f > 0 \) on \( \{v \in T : r \leq v_1 < \overline{x}\} \), the event where the types of all buyers belong to this set will occur with positive probability which implies that with some strictly positive probability a given bidder with \( v_1 < \overline{x} \) wins the first auction and pay at least \( \overline{b} \) for this first unit. The payoff from the first auction [resp. the second auction] of such a buyer winning in those conditions is then at most \( v_1 - \overline{b} \) [resp. \((v_2 - r) \lor 0\)]. Since \( v_2 \leq v_1 < \overline{x} \), we obtain that the total expected payoff is strictly smaller than zero. On the whole, we obtain that such a buyer would have been strictly better off if he has bid \( r \) (or \(-1\)) instead of \( \beta(v) \) which raises a contradiction. Finally, we have shown that \( \beta(v) = r \) if \( v_1 = r \) and thus that \( \overline{b} = r \). Note that we have then \((r, \overline{b}) \subseteq S^* \subseteq S = [r, \overline{b}] \). Since \( G(-1) = G(r) = F1(r) \), we have \( G^1(r) = 0 \) if and only if \( T = T_r \). We conclude then that \( S^* = [r, \overline{b}] \) if \( T \neq T_r \) [resp. \( S^* = (r, \overline{b}) \) if \( T = T_r \).
C Some elements on the conditional expectations

Below we give some technical details that are skipped in the body of the paper. The main aim is to show where our first-order conditions come from.

Consider a bid \( b \in \mathcal{S}^{NA} \) and \( \epsilon \neq 0 \) small enough so that \( G(b + \epsilon) > 0 \). From Lemma 2.2, we have also \( G(b + \epsilon) - G(b) > 0 \). We obtain then that

\[
F_i(x | I(b, \epsilon)) = \frac{G(b + \epsilon)}{G(b)} F_i(x | I(b, \epsilon)) - \frac{G(b + \epsilon) - G(b)}{G(b)} F_i(x | I(b, \epsilon)) + F_i(x | I(b, \epsilon)).
\]

Since \( b \in \mathcal{S}^{NA} \) we have also \( \frac{\partial F_i(x | I(b, \epsilon))}{\partial b} = \lim_{\epsilon \to 0} \frac{F_i(x | I(b, \epsilon)) - F_i(x | I(b, \epsilon))}{\epsilon} \) for any point \( x \in [0, \infty) \) except possibly on a finite number of points, \( G(b) = \lim_{\epsilon \to 0} G(b + \epsilon) \) and \( g(b) = \lim_{\epsilon \to 0} \frac{G(b + \epsilon) - G(b)}{\epsilon} > 0 \). The limit \( \lim_{\epsilon \to 0} F_i(x | I(b, \epsilon)) \) is thus well-defined, it is equal to \( F_i(x | b) \) and from (1) we obtain finally

\[
F_i(x | b) = \frac{G(b)}{g(b)} \cdot \frac{\partial F_i(x | I(b, \epsilon))}{\partial b} + F_i(x | I(b, \epsilon)).
\]

for any \( x \in [0, \infty) \) except possibly on a finite number of points.

Take \( b \in \mathcal{S}^* \). Let \( b_0^n > \ldots > b_l^n \) the atoms in the set \( \{b, b_0^n = b \} \) and \( b_{l+1} = b \). For \( m = 1, \ldots, l + 1 \), let \( E(b, m) := \{ v \in T : b_{m-1} < \beta(v) < b_m \} \). We have then

\[
F_i(x | \beta \leq b) = F_i(x | I(b, \epsilon)) - \frac{1}{G(b)} \frac{G(b)}{G(b) + \epsilon} \left( \sum_{m=0}^{l+1} F_i(x | I(b, \epsilon)) \frac{G(b_m)}{G(b)} + \sum_{m=1}^{l+1} F_i(x | E(b, m)) \frac{G(b_{m-1})}{G(b)} \right). 
\]

(31)

In order to show that \( F_i(x | \beta \leq b) = F_i(x | I(b, \epsilon)) \), then similarly \( F_i(x | \beta < b) = \int_{-1}^{b} F_i(x | u) dE(b, m) \) and it is then sufficient to check that \( F_i(x | E(b, m)) = \int_{-1}^{b} F_i(x | u) \frac{dE(b, m)}{G(b)} \) for any \( m = 1, \ldots, l + 1 \). This comes from the following decomposition which holds for any \( k \in \mathbb{N} \):

\[
F_i(x | E(b, m)) = \sum_{s=1}^{k} F_i(x | I(b_{m-1}^a - b_{m-1}^a) + \frac{(s-1)}{k} (b_m^a - b_{m-1}^a), (b_m^a - b_{m-1}^a)) \cdot \frac{G(b_{m-1}^a + \frac{(s-1)}{k} (b_m^a - b_{m-1}^a)) - G(b_{m-1}^a + \frac{(s-1)}{k} (b_m^a - b_{m-1}^a))}{G(b_{m-1}^a)} \cdot \frac{G(b_{m-1}^a + \frac{(s-1)}{k} (b_m^a - b_{m-1}^a)) - G(b_{m-1}^a + \frac{(s-1)}{k} (b_m^a - b_{m-1}^a))}{G(b_{m-1}^a)}.
\]

For any \( b, \epsilon \in \mathbb{R} \) such that \( I(b, \epsilon) \in \Delta(T) \), let \( H_1(\cdot | I(b, \epsilon)) \) [resp. \( H_2(\cdot | I(b, \epsilon)) \)] denote the CDF of the highest competing bid in the second auction for a given bidder conditional on having won [resp. lost] the first auction with a bid strictly above \( b \land (b + \epsilon) \) [resp. below \( b \land (b + \epsilon) \)] while his highest competing bid was in the set \( [b \land (b + \epsilon), b \lor (b + \epsilon)] \).

Let \( V = (v_1, \ldots, v^{N-1}) \in T^{N-1} \) denote the vector of all types except one and let
Let $\beta_i$ be any event for $1 \leq j \leq k$ and $\beta(v^j) \in [b \land (b + \epsilon), b \lor (b + \epsilon)]$ for $k+1 \leq j \leq N-1$. Let $p_k(b, \epsilon)$ denote the probability of this event conditional on the event $I_{I_{V}^{\max}(b, \epsilon)}$: we have

$$p_k(b, \epsilon) = \frac{\max_{j=1}^{k-1} \beta(v^j)^k (G(b \land (b + \epsilon)) - G(b \land (b + \epsilon)))^{N-1-k}}{\max_{j=1}^{k-1} \beta(v^j)^k (G(b \land (b + \epsilon)) - G(b \land (b + \epsilon)))^{N-1-k}}.$$ Note that we have $\sum_{k=0}^{N-2} (N-1)^k p_k(b, \epsilon) = 1$. Let $j^*(V)$ denote the largest element in the set $\arg\max_{j \in \{1, \ldots, N-1\}} \beta(v^j)$.\[44\] For any $b \in (b, \bar{b})$ and $\epsilon \neq 0$ close enough to zero so that $G(b + \epsilon) > 0$ which guarantees then that $I(b, \epsilon) \in \Delta(T)$ and $\text{Prob}(I_{I_{V}^{\max}}(b, \epsilon)) > 0$ for any $k \in \{0, \ldots, N-2\}$, we have:

$$H_i(x|I(b, \epsilon)) = \mathbb{E}\left[1[v_i^*(V) \leq x] \prod_{j=1}^{N-1} 1[v_j^* \leq x] | V \in I_{I_{V}^{\max}}(b, \epsilon) \right] \cdot F^{\epsilon}(x)$$

and then by independence of the variables $v_j$, $j = 1, \ldots, N-1$, we obtain that

$$H_i(x|I(b, \epsilon)) = \left(\sum_{k=0}^{N-2} \binom{N-1}{k} p_k(b, \epsilon) \cdot \mathbb{E}\left[1[v_i^*(V) \leq x] \prod_{j=1}^{N-1} 1[v_j^* \leq x] | V \in I_{I_{V}^{\max}}(b, \epsilon) \right] \right) \cdot F^{\epsilon}(x)$$

Note also that for $k = N-2$, we have always $j^*(V) = N-1$ conditional on the event $I_{I_{V}^{\max}}(b, \epsilon)$ and $\mathbb{E}\left[1[v_i^*(V) \leq x] \prod_{j=k+1}^{N-1} 1[v_j^* \leq x] | V \in I_{I_{V}^{\max}}(b, \epsilon) \right] = F_i(x|I(b, \epsilon))$. If $b \notin S^A$, then $G(b \lor (b + \epsilon)) - G(b \land (b + \epsilon))$ goes to zero once $\epsilon$ goes to zero. Consequently, for $k = 0, \ldots, N-3$, the coefficient $p_k(b, \epsilon)$ goes to zero when $\epsilon$ goes to zero while $p_{N-2}(b, \epsilon)$ goes to $\frac{1}{N-2}$. On the whole, for $b \in S^* \setminus S^A$ we obtain:

$$\limsup_{\epsilon \to 0, \epsilon \neq 0} H_i(x|I(b, \epsilon)) = [F_i(x|\beta < b)]^{N-2} \cdot F_i(x|b) \cdot F^{\epsilon}(x).$$

For $b \in S^* \setminus S^A$, we have $F_1(x|\beta < b) = \int_{-1}^{b} F_1(x|u) \frac{dG(u)}{G(b)} = \int_{-1}^{b} F_1(x|u) \frac{dG(u)}{G(b)} = F_1(x|\beta \leq b)$ and we obtain finally that the way we have defined $H_i(|b)$ in (3) guarantees that $H_i(x|b) = \limsup_{\epsilon \to 0} H_i(x|I(b, \epsilon))$.

For any $k \in \{0, \ldots, N-2\}$ and $b \in S^A$ we have $E\left[1[v_i^*(V) \leq x] \prod_{j=k+1}^{N-1} 1[v_j^* \leq x] | I_{I_{V}^{\max}}(b, 0) \right] = [F_i(x|b)]^{N-1-k} \cdot F_i(x|b)$. Then we let $\tilde{p}_k(b) := \frac{[G(b)]^k [G(b) - G(0)]^{N-1-k}}{[G(b)]^{N-1} - [G(0)]^{N-1}}$. On
the whole, if \( b \in S^A \cap S^* \) we obtain:

\[
H_i(x|b) = \frac{\sum_{k=0}^{N-2} \binom{N-1}{k} \hat{p}_k(b) \cdot [F_1(x|\beta < b)]^k \cdot [F_1(x|b)]^{N-2-k} \cdot F_{ex}(x)}{\sum_{k=0}^{N-2} \binom{N-1}{k} \hat{p}_k(b)}
\]  

(32)

where \( \sum_{k=0}^{N-2} \binom{N-1}{k} \hat{p}_k(b) = 1 \). In this case note that we have also \( H_i(x|b) = \lim_{\epsilon \to 0} H_i(x|I(b, \epsilon)) \).

If \( b \in S^A \setminus S^* \) (this may occur either for \( b = -1 \) or for \( b = r \)), then we have \( H_i(x|b) = [F_1(x|b)]^{N-2} \cdot F_i(x|b) \cdot F_{ex}(x) \).

Similar calculations leads to:

\[
H_i^*(x|b) = \frac{\sum_{k=0}^{N-2} \binom{N-1}{k} \hat{p}_k(b) \cdot [F_1(x|\beta < b)]^k \cdot [F_1(x|b)]^{N-2-k} \cdot F_{ex}(x)}{\sum_{k=0}^{N-2} \binom{N-1}{k} \hat{p}_k(b)}
\]

(33)

if \( b \in S^A \cap S^* \) and \( H_i^*(x|b) = [F_1(x|b)]^{N-1} \cdot F_{ex}(x) \) if \( b \in S^A \setminus S^* \), where we should interpret the weights \( \frac{1}{N-k} \) (for \( k = 0, \ldots, N-2 \)) in (33) as the probability to win the tie when \( N - k \) bidders are involved in the tie. Note that \( \sum_{k=0}^{N-2} \frac{1}{N-k} \binom{N-1}{k} \hat{p}_k(b) = p_w(b) \).

\[
H_2^*(x|b) = \frac{\sum_{k=0}^{N-2} \frac{N-1-k}{N-k} \binom{N-1}{k} \hat{p}_k(b) \cdot [F_1(x|\beta < b)]^j \cdot [F_1(x|b)]^{N-2-j} \cdot F_2(x|b) \cdot F_{ex}(x)}{\sum_{k=0}^{N-2} \frac{N-1-k}{N-k} \binom{N-1}{k} \hat{p}_k(b)}
\]

(34)

if \( b \in S^A \cap S^* \), \( H_i^*(x|b) = [F_1(x|b)]^{N-2} \cdot F_2(x|b) \cdot F_{ex}(x) \) if \( b \in S^A \setminus S^* \) and \( H_i^*(x|b) = [F_1(x|b)]^{N-1} \cdot F_{ex}(x) \) if \( b = -1 \in S^A \), where we should interpret the weights \( \frac{N-1-k}{N-k} \) (for \( k = 0, \ldots, N-2 \)) in (34) as the probability to lose the tie when \( N - k \) bidders are involved in the tie. Note that \( \sum_{k=0}^{N-2} \frac{N-1-k}{N-k} \binom{N-1}{k} \hat{p}_k(b) = 1 - p_w(b) \). For \( N = 2 \), we have obviously \( H_2^*([1]=b) = H_i([1]=b) \).

From the rule of the auction game, the expected payoff function \( U(v, b) \) is given by

\[
U(v, b) = [(v_1 - r) + \Pi_2^*(v_2|v_1)] \cdot G^{N-1}(r) + \int \int \left[ (v_1 - \beta(v^{(V)})) + ((v_2 - N-1 \max_{i=1}^{N-1} v_i^V) \cup u) \cup 0 \right] 1[r \leq \beta(v^{(V)}) < b] f_V(V) dV dF_{ex}(u) + U_{tie}(v, b) \cdot [G^{N-1}(b) - G^{N-1}(b)] + \int \int \left[ (v_1 - \max_{i=1}^{N-1} v_i^V) \cup v_2^V \cup u \right] \cap \left[ \max_{i=1}^{N-1} v_i^V \geq b \right] f_V(V) dV dF_{ex}(u)
\]

(35)

if \( b \geq r \) and

\[
U(v, -1) = \Pi_2^*(v_2|-1) \cdot G^{N-1}(r) + \int \int \left[ (v_1 - \max_{i=1}^{N-1} v_i^V) \cup v_2^V \cup u \right] \cap \left[ \max_{i=1}^{N-1} v_i^V \geq r \right] f_V(V) dV dF_{ex}(u).
\]

(36)

Note that the third term in (35) is null if \( b \notin S^A \). To see why (35) [resp. (36)] is equivalent to (5) [resp. 6], let us detail a bit why the fourth term in (35) is equal to
\[ \int_{(b)} \Pi_2(v_1|u) \cdot dG^{N-1}(u) \] (the proof for the second term is similar). To avoid additional complications, we assume implicitly below that \([b_1, b_2] \cap S^A = \emptyset \) and \(G(b) > 0\). We can deal with atoms with a similar decomposition as the one used in (31).

\[
\int \int \left[ (v_1 - \frac{\max_{i=1}^{N-1}}{i \neq j} \{v_i^j(V) \vee u\} \vee 0) \cdot 1[|\beta(v^j(V))| > b] f_V(V) dV dF^{\pi}(u) = \right.
\]
\[
= \sum_{s=1}^{k} \int \left[ (v_1 - \frac{\max_{i=1}^{N-1}}{i \neq j} \{v_i^j(V) \vee u\} \vee 0) dF^{\pi}(u) \right] 1[\max_{i=1}^{N-1} \{\beta(v^i)\} \in I(b + \frac{(s - 1)}{k} (b - b), \frac{b - b}{k})] f_V(V) dV
\]
\[
= \lim_{k \to \infty} \left( \sum_{s=1}^{k} \int_{0}^{v_1} (v_1 - x) d \left[ H_2(x|I(b + \frac{(s - 1)}{k} (b - b), \frac{b - b}{k})) \right] \right)
\[
\cdot \left[ G^{N-1}(b + \frac{s}{k} (b - b)) - G^{N-1}(b + \frac{(s - 1)}{k} (b - b)) \right]
\]
\[
= \int_{(b)} \int_{0}^{v_1} (v_1 - x) dH_2(x|u) \cdot dG^{N-1}(u) = \int_{(b)} \Pi_2(v_1|u) \cdot dG^{N-1}(u).
\]

(D) Proof of Lemma 2.3

Since \(H_i(\cdot|b)\) is a CDF and \(x \to (x - u) \vee 0\) is continuously nondecreasing, the function \(\Pi_i(\cdot|b)\) is continuously nondecreasing. If \(x \leq r\), \(\Pi_2(x|b) = 0\). For \(x \geq r\), \(x - \Pi_2(x|b)\) can be written as \(x \cdot (1 - H_2(x|b)) + \int_{0}^{x} udH_2(u|b)\). Consider \(x' \geq x \geq r\). We have \(\int_{x'}^{x} udH_2(u|b) \geq x \cdot (H_2(x'|b) - H_2(x|b)) = x \cdot (1 - H_2(x|b)) - x \cdot (1 - H_2(x'|b)) \geq x \cdot (1 - H_2(x|b)) - x' \cdot (1 - H_2(x'|b))\).

We obtain finally that \(x - \Pi_2(x|b)\) is continuously nondecreasing.

Take a sequence \((b_n)_{n \in \mathbb{N}}\), with \(b_n \in S^*\), which converges to \(b^* \in S^{NA}\) once \(n\) goes to infinity. Since \(b^* \in S^{NA}\), we have \(\lim_{n \to \infty} G(b_n) = G(b^*) > 0\) and \(\lim_{n \to \infty} g(b_n) = g(b^*) > 0\). Except on a finite number of points \(x\), \(b \to F_i(x|\beta \leq b)\) is continuously differentiable at \(b^*\) (regularity assumption). From (2), we obtain then that \(F_i(\cdot|b_n)\) converges weakly to \(F_i(\cdot|b^*)\). From the definition of \(H_i(\cdot|b^*)\), we obtain then that \(H_i(\cdot|b_n)\) converges weakly to \(H_i(\cdot|b^*)\). From portmanteau theorem and since \(u \to (x - u) \vee 0\) is a continuous function, this implies that \(\lim_{n \to \infty} \Pi_i(x|b_n) = \Pi_i(x|b^*)\) for any \(x\). On the whole, we have shown that the function \(b \to \Pi_i(x|b)\) \((i = 1, 2)\) is continuous at any point in \(S^{NA}\). Furthermore, we have \(0 \leq \Pi_i(x|b) \leq x\) for any \(b \in S^* \cup S^A\). We conclude that for any given \(x\), the function \(b \to \Pi_i(x|b)\) \((i = 1, 2)\) is uniformly bounded (on \(S^*\)).
E Proof of Lemma 3.3

Take \( v' \in L^*(v) \) [resp. \( v' \in M^*(v) \)]. Suppose now that for any \( \epsilon > 0 \), there exists \( v'' \in \mathcal{P} \) such that \( v'_1 \leq v''_1 \) [resp. \( v''_2 \geq v'_2 \)] and \(|\beta(v'') - b| \leq \epsilon \). Since \( \beta \) is continuous on the set \( \mathcal{P} \) which is compact, we obtain then that there exists \( \tilde{v} \in \mathcal{P} \) such that \( \tilde{v}_1 \leq v''_1 \) [resp. \( \tilde{v}_2 \geq v''_2 \)] and \( \beta(\tilde{v}) = b \). Since \( v' \in L^*(v) \) [resp. \( v' \in M^*(v) \)], this implies that \( \tilde{v} \in L^*(v) \) [resp. \( \tilde{v} \in M^*(v) \)] which raises a contradiction with Lemma 3.2. There exists thus \( \epsilon > 0 \) such that for all \( \tilde{v} \in \mathcal{P} \) with \( \tilde{v}_1 \leq v''_1 \) [resp. \( \tilde{v}_2 \geq v''_2 \)] and all \( v' \) with \(|v'| < \epsilon \), we have \( \tilde{v} \notin I(b,e') \). This implies that \( F_1(v'_1[I(b,e')]) = 0 \) [resp. \( F_2(v''_2[I(b,e')]) = 1 \)] for all \( v' \) with \(|v'| < \epsilon \). Since this holds for any \( v' \in L^*(v) \) [resp. \( v' \in M^*(v) \) and \( F_2((I(b,e')) \) is right-continuous], we obtain then that \( F_1(x[I(b,e')]) = 0 \) for any \( x < v_2 \) [resp. \( F_2(v_2[I(b,e')]) = 1 \)], and then from (1) that \( F_1(x|b) = 0 \) for \( x < v_2 \) [resp. \( F_2(v_2|b) = 1 \)]. From the definition of \( H_1,|b \) (see Appendix C for the formal expression when \( b \in \mathcal{S}^A \)), we obtain finally that \( H_1(x|b) = 0 \) for any \( x < v_2 \) and then as a corollary that \( \Pi_1(v_2|b) = 0 \).

F Proof of Lemma 3.5

We prove the result when \( v_1 = v'_1 \in [r,\overline{v}) \). By continuity, the result extends when \( v_1 = v'_1 = \overline{v} \) and thus for any \( v, v' \in \mathcal{P} \).

Suppose on the contrary that \( \beta(v) \neq \beta(v') \) and w.l.o.g. consider that \( \beta(v') > \beta(v) \). Pick then \( b \in \mathcal{S}^A \cap \{ \beta(v), \beta(v') \} \neq \emptyset \). Since \( \beta \) is continuous on \( \mathcal{L}(v,v') \), there exists \( \tilde{v} \in \mathcal{L}(v,v') \) (which guarantees that \( \tilde{v}_1 = v_1 \) such that \( \beta(\tilde{v}) = b \). There exists also \( \alpha > 0 \) such that \( \beta(\tilde{v}) < b \) for any \( \tilde{v} \in Z(v,\alpha) \). Furthermore, since \( v \in \mathcal{P} \) and \( v_1 < \overline{v} \), we have \( Z(v,\alpha) \cap \{ \tilde{v} \in \mathcal{T} : \tilde{v}_1 > v_1 \} \in \Delta(\mathcal{T}) \). There is thus a positive measure of types whose first valuations are strictly above \( v_1 \) and that bid strictly less than \( b \) or equivalently \( F_1(v_1|\beta < b) < 1 \). Pick thus \( \tilde{v} \in Z(v,\alpha) \cap \{ \tilde{v} \in \mathcal{T} : \tilde{v}_1 > v_1 \} \) such that \( \tilde{v} \neq v_1 \). Since \( \beta(v') > b > \beta(\tilde{v}) \), there exists \( v'' \in \mathcal{L}(v',\tilde{v}) \) such that \( \beta(v'') = b \). Since \( v'' \in \mathcal{L}(v',\tilde{v}) \), we have finally \( v''_1 > \tilde{v}_1 = v_1 \). By applying Lemma 3.4 to the pair of types \( \hat{v}, v'' \) and \( b = \beta(\hat{v}) = \beta(v'') \in \mathcal{S}^A \), we obtain that \( F^{ex}(v_1) = 1 \) and that \( F_1(v_1|\beta < b) = 1 \) if \( N \geq 3 \). On the whole, we have raised a contradiction either if \( F^{ex}(v_1) < 1 \) or if \( N \geq 3 \), which concludes the proof.

G Proof of Lemma 3.6

Suppose on the contrary that there exists \( v \in L^*(\overline{v'}) \cap M^*(\overline{v}) \) such that \( \beta(v) \neq b \).

Consider first the case \( N = 2 \). From Lemma 3.1, if \( \beta(v) < b \) [resp. \( \beta(v) > b ] \), then \( v \in M^*(\overline{v}) \) [resp. \( v \in L^*(\overline{v}) \)] implies that \( b \in S_{U}^{max}(v) \). From Lemma 3.2, we obtain that
$b \notin \mathcal{S}^A$. Since $\beta$ is continuous, there exists $\alpha > 0$ such that $\beta(v) < \beta(\tilde{v})$ and $\tilde{v} \in L^*(v)$ for any $\tilde{v} \in Z(\tilde{v}, \alpha)$ [resp. $\beta(v) > \beta(\tilde{v})$ and $v \in L^*(\tilde{v})$ for any $\tilde{v} \in Z(\tilde{v}, \alpha)$]. Furthermore, since $b \notin \mathcal{S}^A$, then there exists $\tilde{v} \in Z(\tilde{v}, \alpha)$ [resp. $\tilde{v} \in Z(\tilde{v}, \alpha)$] such that $\beta(\tilde{v}) \in \mathcal{S}^{NA}$. Applying the same argument as above, we obtain from Lemma 3.1 that $\beta(\tilde{v}) \in S_U^{\max}(v)$. If we apply Lemma 3.2 to $\tilde{v}$, $v$ and $\beta(\tilde{v}) \in \mathcal{S}^{NA}$, we raise then a contradiction.

Consider now the case $N \geq 3$. First, we obtain from Lemma 3.5 that $\hat{\beta}(v_1) = \hat{\beta}(v_1') = b$ and $\beta(v) = \hat{\beta}(v_1)$. Second, if $\beta(v) < b$ [resp. $\beta(v) > b$], then by continuity we can pick $\tilde{v} \in \mathcal{L}((v_1, \tilde{v}_1), (v_1, v_1))$ such that $\beta(\tilde{v}) < b$ [resp. $\tilde{v} \in \mathcal{L}((v_1', \tilde{v}_1'), (v_1, v_1))$ such that $\beta(\tilde{v}) > b$] and $\beta(\tilde{v}) \in \mathcal{S}^{NA}$. We have also $(\tilde{v}_1, \tilde{v}_1) \in L^*(\tilde{v})$ [resp. $\tilde{v} \in L^*((\tilde{v}_1', \tilde{v}_1'))$. From Lemma 3.1, we obtain then that $\beta(\tilde{v}) \in S_U^{\max}((\tilde{v}_1, \tilde{v}_1'))$ [resp. $\beta(\tilde{v}) \in S_U^{\max}((\tilde{v}_1', \tilde{v}_1'))$]. If we apply Lemma 3.2 to $\tilde{v}$, $(\tilde{v}_1, \tilde{v}_1)$ [resp. $(\tilde{v}_1', \tilde{v}_1')$] and $\beta(\tilde{v}) \in \mathcal{S}^{NA}$, we raise then a contradiction.

We have thus shown that $\beta(v) = b$ for any $v \in L^*(\tilde{v}') \cap M^*(\tilde{v})$. We have then $G(b) - G(b) \geq \int \int [v \in L^*(\tilde{v}') \cap M^*(\tilde{v})] f(v) dv$. If $\tilde{v}' \gg \tilde{v}$, then $L^*(\tilde{v}') \cap M^*(\tilde{v}) \in \Delta(T)$ which implies then that $b \in \mathcal{S}^A$. If $N = 2$, then we can apply Lemma 3.2 to $\tilde{v}', \tilde{v} \in \mathcal{P}$ and $b \in \mathcal{S}^A$. We obtain then that $\tilde{v}' \notin M^*(\tilde{v})$ which raises a contradiction with our assumption $\tilde{v}' \gg \tilde{v}$. On the whole we obtain that $N \geq 3$.

H Proof of Lemma 3.7

Suppose that $\hat{\beta}(x) > \hat{\beta}(x')$ for $x, x'$ such that $r < x < x' \leq \overline{r}$. Since $\hat{\beta}(r) = r$ (Lemma 2.2), we obtain by continuity that there exists $x'' \in [r, x]$ such that $\hat{\beta}(x'') = \hat{\beta}(x')$. From Lemma 3.6, we obtain that $\hat{\beta}(u) = \hat{\beta}(x')$ for any $u \in [x'', x']$ and thus for $u = x$ which raises a contradiction. On the whole we have shown that $\hat{\beta}$ is nondecreasing.

From Lemma 3.6, we obtain that if $N = 2$, then $\hat{\beta}(x) = \hat{\beta}(x')$ for $x, x' \in [r, \overline{r}]$ implies that $x = x'$. $\hat{\beta}$ is thus strictly increasing if $N = 2$. Consider now $N \geq 3$. Suppose that there exists $x, x' \in [r, \overline{r}]$ with $x' > x$ and $\hat{\beta}(x) = \hat{\beta}(x') = b$. Let $\underline{x} := \inf\{u \in [r, \overline{r}] : \hat{\beta}(u) = b\}$ and $\overline{x} := \sup\{u \in [r, \overline{r}] : \hat{\beta}(u) = b\}$. We have $\overline{x} > \underline{x}$. From Lemma 3.5 and since $\hat{\beta}$ is nondecreasing, we obtain that $\beta(v) = b$ if and only if $v_1 \in [\overline{r}, \underline{x}]$. Since $\overline{x} > \underline{x}$, we have then $b \in \mathcal{S}^A$ with $G(b) - G(b) = F1(\overline{x}) - F1(\underline{x}) > 0$, $p_w(b) < \frac{1}{2}$ and $F1(\underline{x}|b) = 0$. From (32) and (34), we have then $H_2(u|b) = \frac{N-1}{N-2} p_{N-2}(b) \cdot [F1(u|\beta < b)]^{N-2} \cdot F2(u|b) \cdot F^{\sigma}(u)$ and $H_2^*(u|b) = \frac{1}{1-p_w(b)} \cdot [F1(u|\beta < b)]^{N-2} \cdot F2(u|b) \cdot F^{\sigma}(u)$ for any $u \leq \underline{x}$. We obtain then that $\Pi_2(\underline{x}|b) = 2(1 - p_w(b)) \cdot \Pi_3^*(\underline{x}|b)$. Note also that $U_{tie}(v, b) = p_w(b) \cdot [v_1 - b] + (1 - p_w(b)) \cdot \Pi_2^*(v_1|b)$ for any $v \in \mathcal{T}$ with $v_2 \leq r$.

Consider first the case where $b > r$. If we apply the equilibrium conditions (8) and (9) for $v = (\underline{x}, 0)$ and $b = \beta((\underline{x}, 0))$, we obtain then respectively $\frac{1}{2} \cdot \Pi_2(\underline{x}|b) \geq (1 - p_w(b)) [\underline{x} - b]$. 

47
and \( p_w(b) \cdot [x - b] \geq \frac{1}{2} \cdot \Pi_2(x|b) \). From the second inequality and since \( \Pi_2(x|b) \geq 0 \), we obtain that \( b \leq x \). Combining the two inequalities and since \( p_w(b) < \frac{1}{2} \), we obtain that \( b \geq x \). On the whole we have \( b = x = r \) and \( \Pi_2(x|b) = 0 \). Conditional on the highest bid from the opponents being equal to \( b \) and that one those bidders win the first auction, there is a strictly positive probability that all the bids from those initial bidders are below \( \frac{r + x}{2} \) in the second stage. \( \Pi_2(x|b) = 0 \) implies thus that \( \beta^{ex} = x \) which guarantees that \( \Pi_2^*(\tilde{x}|b) \leq \tilde{x} - x \) for any \( \tilde{x} \geq x \). However, conditional on the highest bid from the opponents being equal to \( b \) and that one those bidders win the first auction, there is a strictly positive probability that at least one bid from those initial bidders is strictly above \( x \) in the second stage (in particular because \( F_1(x|b) = 0 \)) and we have thus \( \Pi_2^*(\tilde{x}|b) < \tilde{x} - x \) for any \( \tilde{x} > x \). We obtain finally that \( U_{tie}(\pi, 0, b) < \pi - x = \pi - b \) which raises a contradiction with the equilibrium condition (8).

Consider then the case where \( b = r \) which implies also that \( x = r \) (Lemma 2.2). Conditional on the highest bid from the opponents being equal to \( r \) and that one those bidders win the first auction, there is a strictly positive probability that at least one bid from those initial bidders is strictly above \( r \) in the second stage (in particular because \( F_1(r|b) = 0 \)) and we have thus \( \Pi_2^*(\tilde{x}|b) < \tilde{x} - r \) for any \( \tilde{x} > r \). We obtain finally that \( U_{tie}(\pi, 0, b) < \pi - r \) which raises a contradiction with the equilibrium condition (8). We have thus shown that \( \tilde{\beta} \) is strictly increasing if \( N \geq 3 \). In any case, we obtain also as a corollary that \( \tilde{\beta} > r \).

It remains to show that \( \tilde{\beta} = \hat{\beta}(\pi) \). Suppose on the contrary that \( \tilde{\beta} > \hat{\beta}(\pi) \). Then there exists \( v \in L^*((\pi, \pi')) \) such that \( \beta(v) > \hat{\beta}(\pi) \). Since \( \hat{\beta}(r) = r \) and \( \beta \) is continuous, we obtain then that there exists \( v' \in L((r, r), v) \subseteq L^*((\pi, \pi')) \) such that \( \beta(v') = \hat{\beta}(\pi) \). From Lemma 3.6, we obtain that \( \hat{\beta} \) is constant on \( (v', \pi] \) which raises a contradiction with \( \hat{\beta} \) is strictly increasing and thus concludes the proof.

I Proof of Lemma 3.8

Parts 1) and 2) come directly from the definitions of the RP and WM properties. Exactly as in the proof of Lemma 3.3, the first part of the WM property implies that for any \( b \in S \), we have \( F_1(x|b) = 0 \) if \( x < \hat{\beta}^{-1}(b) \) and \( F_2(x|b) = 1 \) if \( x \geq \hat{\beta}^{-1}(b) \). The second part of the WM property implies that \( F_1(x|\beta < b) = \frac{F_1(x|\hat{\beta}^{-1}(b))}{F_1(\hat{\beta}^{-1}(b))} \) if \( b \in S^* \) and \( N \geq 3 \). Note finally that \( F^{ex}(x) = 0 \) for any \( x < \tilde{b}^{ex} \). Gathering the previous elements, we observe first that \( H_1(x|b) = 0 \) for any \( b \in S^* \cup S^\perp_1 \) and \( x < \hat{\beta}^{-1}(b) \vee \tilde{b}^{ex} \) and then part 3), and second that \( H_2(x|b) = F^{ex}(x) \) for any \( b \in S^* \cup S^\perp_1 \) and \( x \geq \hat{\beta}^{-1}(b) \) and thus part 4).
J Proof of Lemma 3.9

From Lemma 3.5, we obtain that $\beta(v) = \hat{\beta}(v_1)$ if $v_1 < x^*$. By continuity, this is also true if $v_1 = x^*$.

Consider $v \in T \setminus L((x^*, x^*))$ (which implies that $x^* < v$ and thus $N = 2$) and suppose that $\beta(v) < \hat{\beta}(x^*)$. Since $\beta$ is continuous, we can suppose w.l.o.g. that $\beta(v) \in S^{NA} \cup S^A_+$. From Lemma 3.7, we obtain that there exists also $x \in [r, x^*)$ such that $\beta(v) = \hat{\beta}(x)$. From Lemma 3.4 and since $v_1 > x$, we obtain then that $F^{1x}(x) = 1$ which raises a contradiction with $x < x^* = \hat{\beta}(x)$. On the whole, we obtain that for any $v \in T \setminus L((x^*, x^*))$, then we have $\beta(v) \geq \hat{\beta}(x^*)$.

We obtain finally that $\beta(v) < \hat{\beta}(\gamma)$ since $F^{1} \text{ contains no atoms}$ on the interval $[r, \hat{\beta}(x^*)]$ since $F^1 \text{ contains no atoms}$. From the WM property and Lemma 3.7, we have $\beta(v) < \hat{\beta}(\gamma)$ for any $v \in L^*([\gamma, \gamma])$. Since $T \setminus L^*([\gamma, \gamma]) \notin \Delta(T)$, we obtain that $\hat{\beta} \notin S^A$.

To conclude the proof, we have to check that $\beta$ is given by (17) on $T \cap L((x^*, x^*))$. By continuity, it is sufficient to check that it is true on $T \cap L^*((x^*, x^*))$. For $b \in S^A$ with $b < \hat{\beta}(x^*)$, we have then: $F_1(x|\beta < b) = \frac{F_1(x \wedge \hat{\beta}^{-1}(b))}{F_1(\hat{\beta}^{-1}(b))}$ and $F_2(x|b) = \lim_{\epsilon \to 0} \int_{v_1 \in \hat{\beta}^{-1}(b), x} f(v)dv$.

$$\lim_{\epsilon \to 0} \int_{v_1 \in \hat{\beta}^{-1}(b), x} \frac{f(v)v_1}{f(v)dv} = \int_{v_1 \in \hat{\beta}^{-1}(b), x} f(v)dv = F_2(x|\hat{\beta}^{-1}(b)).$$

For any $v \in T \cap L^*((x^*, x^*))$ such that $b = \beta(v) \in S^{NA}$ (and thus for almost all types in $T \cap L^*((x^*, x^*))$ since $\beta(v) \notin S^A_+$ for any $v \in T \cap L^*((x^*, x^*))$), the first-order condition (14) holds and is equivalent to (17). Since the right-hand side of (17) is continuous in $v_1$ whereas we have assumed that $\beta$ is continuous, we obtain finally that (17) holds for all types in $T \cap L^*((x^*, x^*))$.

K Proof of Proposition 3.3

We show actually a slightly stronger submartingale property: for any $v^{(2;N)}$ such that $p^1 = \beta(v^{(2;N)}) \in S^{NA} \cup S^A_+ \cup \{r\}$, we have

$$E[p^2|v^{(2;N)}, p^1] \geq p^1 \quad (38)$$

for any $p^1 \in S^{NA} \cup S^A_+ \cup \{r\}$ where $p^1$ denotes the highest bid at the first stage, i.e. $p^1 = \beta(v^{(1;N)}) \geq p^1$. The submartingale property comes then by iterated expectations.\footnote{In particular note that conditional on $p^1$, the variable $v^1$ is distributed such that $v^1 \in S^{NA} \cup S^A_+ \cup \{r\}$ with probability one. E.g., if $p^1 \notin S^A \cup \{r\}$, then by independence of types and thus of the bids, $p^1$ is distributed according to the CDF $b \rightarrow \frac{G(b \mid v^1=p^1)-G(b)}{1-G(b)}$.}

Note that (38) holds straightforwardly if $p^1 = r$ (and so a fortiori if $p^1 = r$). Next we
assume then that \( p^{1}, \overline{p}^{1} \in S^{N_{A}} \cup S_{A}^{1} \).

In a first step, we note that
\[
E \left[ p^{2}|v^{(2:N)}, \overline{p}^{1} \right] = \int_{0}^{\beta} E \left[ p^{2}|v^{(2:N)}, v^{(1:N)}_{2} = u \right] dF_{2}(u|p^{1}) \geq \int_{0}^{\beta} E \left[ p^{2}|v^{(2:N)}, v^{(1:N)}_{2} = u \right] dF_{2}(u|p^{1})
\]
where the first equality results by iterated expectations, the inequality comes from our first-order stochastic dominance assumption combined with the fact that \( u \rightarrow E \left[ p^{2}|v^{(2:N)}, v^{(1:N)}_{2} = u \right] \) is nondecreasing and the last equality comes from \( F_{2}(v^{(2:N)}_{1} | \beta(v^{(2:N)})) = 1 \) which is a consequence of the WM property (see Lemma 3.8).

In a second step, we note that
\[
\int_{0}^{v^{(2:N)}_{1}} E \left[ p^{2}|v^{(2:N)}, v^{(1:N)}_{2} = u \right] dF_{2}(u|p^{1}) = \int_{0}^{v^{(2:N)}_{1}} \left[ v^{(1:N)}_{1} (1 - F^{\text{ex}}(v^{(1:N)}_{1})) \right] dF_{2}(u|p^{1}) = v^{(2:N)}_{1} (1 - F^{\text{ex}}(v^{(2:N)}_{1})) + \int_{0}^{v^{(2:N)}_{1}} udH_{2}(u|p^{1}) = \beta(v^{(2:N)}) = p^{1}
\]
(39)

where the penultimate equality comes from the LM property while the other ones result from our definitions. On the whole we have shown that \( E \left[ p^{2}|v^{(2:N)}, \overline{p}^{1} \right] \geq p^{1} \) which concludes the proof.

K.1 Additional comments on price trends

It is instructive to compare our proof with Katzman’s [25] proof for the equilibrium he exhibited where the bid function is given by (17) for any \( v \in T_{r} \), a bid function that is shown to be an equilibrium also in our generalized environment (Subsection 3.3). Katzman [25] first argues that conditional on the event that the losing buyers at the first stage do not acquire the second unit (so that \( v^{(1:N)}_{2} \geq v^{(2:N)}_{1} \)) then \( p^{2} \geq v^{(2:N)}_{1} \geq \beta(v^{(2:N)}) = p^{1} \), i.e. prices are nondecreasing. Since \( \beta(v) \leq v^{1} \) for any \( v \in P \), this part of Katzman’s [25] argument holds also very generally for any equilibrium in our more general framework.

Second, in order to conclude Katzman [25] shows that
\[
\int_{0}^{v^{(2:N)}_{1}} E \left[ p^{2}|v^{(2:N)}, v^{(1:N)}_{2} = u \right] d \left[ \frac{F_{2}(u|p^{1})}{F_{2}(v^{(2:N)}_{1}|p^{1})} \right] = p^{1}
\]
(40)

for any \( p^{1} \geq p^{1} \). If we integrate this equality over \( p^{1} \), we obtain that conditional on the fact that the winner of the first auction has not the highest valuation among the initial buyers in the last stage (i.e. \( v^{(1:N)}_{2} \leq v^{(2:N)}_{1} \)), the price sequence is a martingale. This property, which can be directly tested in the data once bidders identities are observed, is
actually specific to both the special equilibrium he considers (that we will call later the R1-equilibrium) and the special sampling scheme he uses, which combined together guarantees that \( \frac{F_2(u|p^1)}{F_2(v_1^{(2,N)}|p^2)} = \left( \frac{F_1(u)}{F_1(v_1^{(2,N)})} \right)^{1/2} = \frac{F_2(u|\tilde{\beta}(\beta^{-1}(p^1)))}{F_2(v_1^{(2,N)}|\tilde{\beta}^{-1}(p^1))} = \frac{F_2(u|p^1)}{F_2(v_1^{(2,N)}|p^2)} \) for any \( u \in [0,v_1^{(2,N)}] \). Joint with (39), the equality \( \frac{F_2(u|p^1)}{F_2(v_1^{(2,N)}|p^2)} = \frac{F_2(u|p^1)}{F_2(v_1^{(2,N)}|p^2)} \) implies (40).

We emphasize that this equilibrium prediction is no longer valid in our more general setup, even if we limit ourselves to the R1-equilibrium. Nevertheless, if we make the additional assumption that \( F_2(\cdot,|\cdot) \) is log-supermodular,\(^{46}\) then we obtain that the analog of (40) for the R1-equilibrium is an inequality:

\[
\int_{0}^{v_1^{(2,N)}} E \left[ p^2|v_1^{(2,N)},v_2^{(1,N)} = u \right] \frac{F_2(u|p^1)}{F_2(v_1^{(2,N)}|p^2)} \geq \int_{0}^{v_1^{(2,N)}} E \left[ p^2|v_1^{(2,N)},v_2^{(1,N)} = u \right] \frac{F_2(u|p^1)}{F_2(v_1^{(2,N)}|p^2)} = p^1, \tag{41}
\]

where the inequality results from the log-supermodularity assumption while the equality comes from (39). In other words, we have exhibited a new channel in favor of increasing price patterns: this is the positive correlation between the first and the second valuation of a given bidder. This channel was absent in Katzman [25] due to his special sampling scheme where valuations were coming from two independent draws from a common distribution.

**L Proof of Proposition 3.4**

If \( v_1 < r \), we have already shown in the proof of Lemma 2.1 that \( S_U^{max}(v) = \{-1\} \). Take now \( v \in T \) with \( v_1 \in (r,T) \) and let \( Y := \{ \tilde{\beta} \big( v_2 \lor (b^{ex} \lor v_1) \} \), \( \tilde{\beta}(v_1) \} \).

From Lemma 3.9, we have \( \beta(v) = \tilde{\beta}(v_1) \leq \tilde{\beta}(x^*) \) if \( v_1 \leq x^* \) and \( \beta(v) \geq \tilde{\beta}(x^*) \) otherwise. On the whole, we obtain that \( \beta(v) \geq \tilde{\beta}(x^* \lor v_1) \geq \tilde{\beta}(b^{ex} \lor v_1) \). Combined with part 1) of Lemma 3.8, we obtain that \( \beta(v) \in Y \).

If \( v_1 \leq b^{ex} \), then \( Y = \{ \tilde{\beta}(v_1) \} = \{ \beta(v) \} \) and we have obviously \( Y \subseteq S_U^{max}(v) \). Consider now that \( v_1 > b^{ex} \). We obtain then by definition of \( Y \) that \( \tilde{\beta}^{-1}(b) \geq b^{ex} \) for any \( b \in Y \). First we obtain from Lemma 3.8 that \( \Pi_2(v_1|b) - \Pi_2(\tilde{\beta}^{-1}(b)|b) = v_1 - \tilde{\beta}^{-1}(b) \) and \( \Pi_1(v_2|b) = 0 \) for any \( b \in Y \cap S^* \). Second, if \( b \in S^{NA} \) [resp. \( b \in S^A \)], we have also from (14) [resp. from Remark 3.1] that \( b = \tilde{\beta}^{-1}(b) - \Pi_2(\tilde{\beta}^{-1}(b)|b) \). Combining the previous equalities, we obtain that \( \Pi_2(v_1|b) = (v_1 - b) + \Pi_1(v_2|b) \) for any \( b \in (S^{NA} \cup S^A) \) \( \cap Y \). On the whole this implies that the function \( b \rightarrow U(v,b) \) is constant on the interval \( Y \). Since \( \beta(v) \in Y \), we obtain finally that \( Y \subseteq S_U^{max}(v) \). In order to prove, the inclusion \( \supseteq \) in (18), we are left with showing that \( -1 \in S_U^{max}(v) \) in the case where \( v_2 \leq r = b^{ex} \). First \( v_2 \leq r = b^{ex} \) implies that \( r \in Y \) and we have thus also \( r \in S_U^{max}(v) \). Second, we show next that

\(^{46}\) \( F_2(\cdot,|\cdot) \) log-supermodular means that \( F_2(v_2|v_1^{(i)}) F_2(v_2|v_1) \leq F_2(v_2|v_1) F_2(v_2|v_1^{(i)}) \) once \( v_1^{(i)} \geq v_1 \) and \( v_2 \geq v_1^{(i)} \). This assumption is stronger than conditional stochastic dominance and weaker than affiliation.
\( U(v, -1) = U(v, r) \). Since \( v_2 \leq r \) and \( \overline{b}^{\infty} = r \), we obtain by applying (5) and (6) that
\[
U(v, r) - U(v, -1) = \left[ U_{tie}(v, r) - \Pi_2(v_1|r) \right] \cdot [G^{N-1}(r) - G^{N-1}(\overline{r})] .
\]
If \( r \not\in S^A \), we have then \( U(v, r) = U(v, -1) \). If \( r \in S^A \), this previous equality results from (11) and since \( r \in S^{\text{max}}_U(v) \). On the whole we have thus \(-1 \in S^{\text{max}}_U(v) \) when \( v_2 \leq r = \overline{b}^{\infty} \).

We are left with the inclusion \( \subseteq \) in (18).

1/ Consider first the case \( b > \hat{\beta}(v_1) \) and let us show that \( U(v, b) < U(v, \beta(v)) \). Since \( b > \hat{\beta}(v_1) \), we have
\[
U(v, b) - U(v, \beta(v)) = U(v, b) - U(v, \hat{\beta}(v_1)) = \left[ (v_1 - \hat{\beta}(v_1)) + \Pi_1(v_2|\hat{\beta}(v_1)) - U_{tie}(v, \hat{\beta}(v_1)) \right] \cdot [G^{N-1}(\hat{\beta}(v_1)) - G^{N-1}(\hat{\beta}(v_1))]
+ \int_{(\hat{\beta}(v_1))}^{(b)} [(v_1 - u) - \Pi_2(v_1|u) + \Pi_1(v_2|u)] \cdot dG^{N-1}(u) + [U_{tie}(v, b) - \Pi_2(v_1|b)] \cdot [G^{N-1}(b) - G^{N-1}(\hat{\beta}(v_1))].
\]
(42)

From Remark 3.1, the first term is null since \( \hat{\beta}(v_1) \in S^{\text{max}}_U(v) \) while the third term is null except possibly if \( N = 2 \) and \( b \in S^A_+ \). From Lemma 3.8, we have \( \Pi_1(v_2|u) = 0 \) for any \( u \geq \hat{\beta}(v_2) \) and thus a fortiori if \( u \geq \hat{\beta}(v_1) \). We obtain then
\[
U(v, b) - U(v, \beta(v)) = \int_{(\hat{\beta}(v_1))}^{(b)} [(v_1 - \Pi_2(v_1|u)) - u] \cdot dG^{N-1}(u)
+ \frac{1}{2} \cdot [(v_1 - \Pi_2(v_1|b)) - b] \cdot [G^{N-1}(b) - G^{N-1}(\hat{\beta}(v_1))].
\]
(43)

After applying (14) to the pairs \((u, \hat{\beta}^{-1}(u))\) (which is appropriate since \( u \in S^{\text{max}}_U(\hat{\beta}^{-1}(u)) \) and since atoms arise only for \( N = 2 \)), (43) becomes
\[
U(v, b) - U(v, \beta(v)) = \int_{(\hat{\beta}(v_1))}^{(b)} [(v_1 - \Pi_2(v_1|u)) - (\hat{\beta}^{-1}(u) - \Pi_2(\hat{\beta}^{-1}(u)|u))] \cdot dG^{N-1}(u)
+ \frac{1}{2} \cdot [(v_1 - \Pi_2(v_1|b)) - (\hat{\beta}^{-1}(b) - \Pi_2(\hat{\beta}^{-1}(b)|b))] \cdot [G^{N-1}(b) - G^{N-1}(\hat{\beta}(v_1))].
\]
(44)

Since \( \hat{\beta}^{-1}(u) \geq v_1 \) for any \( u \geq \hat{\beta}(v_1) \), we have from Lemma 2.3 that all the terms in the integral and in the second term are negative. As a corollary, the equality \( U(v, b) = U(v, \beta(v)) \) would imply that \( v_1 - \Pi_2(v_1|u) = \hat{\beta}^{-1}(u) - \Pi_2(\hat{\beta}^{-1}(u)|u) \) for all \( u \in S^A_+ \) with \( u \in (\beta(v), b) \). Let \( u^* \) denote a particular solution. In the same way as in the proof of Lemma 3.2, this implies that \( H_2(x|u^*) = 1 \) for any \( x > v_1 \). But from part 3) of Lemma 3.8, we have also \( H_1(x|u^*) = 0 \) for any \( x < \hat{\beta}^{-1}(u^*) \). For \( x \in (v_1, \hat{\beta}^{-1}(u^*)) \) (which is not empty since \( u^* > \hat{\beta}(v_1) \)), we have thus \( H_1(x|u^*) = 0 < 1 = H_2(x|u^*) \) which raises a contradiction with Remark 2.2. On the whole we have shown that \( U(v, b) < U(v, \beta(v)) \).

2/ Consider now the case \( b < \hat{\beta} \left( v_2 \lor (\overline{b}^{\infty} \land v_1) \right) \) with \( v_2 \lor \overline{b}^{\infty} > r \) and let us show that \( U(v, b) < U(v, \beta(v)) \). Note that \( r < v_2 \lor \overline{b}^{\infty} \) and \( v_1 > r \) implies that \( r < \hat{\beta} \left( v_2 \lor (\overline{b}^{\infty} \land v_1) \right) \).

2a/ In a first step, we consider \( b \in [r, \hat{\beta} \left( v_2 \lor (\overline{b}^{\infty} \land v_1) \right) \) . We have thus \( b < \beta(v) \) and then

52
\[ U(v, \beta(v)) - U(v, b) = \left[(v_1 - b) + \Pi_1(v_2[b] - U_{tie}(v, b)] \cdot [G^{N-1}(b) - \xi^{N-1}(b)] \right. \\
+ \left. \int_{(b)}^{(\beta(v))} [(v_1 - u) - \Pi_2(v_1 | u) + \Pi_1(v_2 | u)] \cdot dG^{N-1}(u) \right. \\
+ \left. [U_{tie}(v, \beta(v)) - \Pi_2(v_1 | \beta(v))] \cdot [G^{N-1}(\beta(v)) - \xi^{N-1}(\beta(v))]. \right) \] (45)

From Remark 3.1, the third term is null while the first term is null except possibly if \( N = 2 \) and \( b \in S_A^1 \). After applying (14) to the pairs \((u, \hat{\beta}^{-1}(u))\) and since atoms arise only for \( N = 2 \), (45) becomes then

\[ U(v, \beta(v)) - U(v, b) = \frac{1}{2} \left[(v_1 - \Pi_2(v_1[b]) - (\beta^{-1}(b) + \Pi_2(\beta^{-1}(b)[b]) + \Pi_1(v_2[b])) \cdot [G^{N-1}(b) - \xi^{N-1}(b)] \\
+ \int_{(b)}^{(\beta(v))} [(v_1 - \Pi_2(v_1 | u)) - (\beta^{-1}(u) - \Pi_2(\beta^{-1}(u) | u)) + \Pi_1(v_2 | u)] \cdot dG^{N-1}(u). \right) \] (46)

For any \( u \leq \hat{\beta}(v_1) \) and thus a fortiori for \( u \leq \beta(v) \), we obtain from Lemma 3.8 a more general version of (15):

\[ \Pi_2(v_1 | u) - \Pi_2(\hat{\beta}^{-1}(u) | u) = v_1 F^{ex}(v_1) - \hat{\beta}^{-1}(u) F^{ex}(\hat{\beta}^{-1}(u)) - \int_{\beta^{-1}(u)}^{v_1} x dF^{ex}(x). \] (47)

Plugging (47) into (46) leads to

\[ U(v, \beta(v)) - U(v, b) = \frac{1}{2} \left[(v_1 - \beta^{-1}(b)) \cdot (1 - F^{ex}(v_1)) + \int_{\beta^{-1}(b)}^{v_1} (x - \beta^{-1}(b)) dF^{ex}(x) + \Pi_1(v_2[b]) \cdot [G^{N-1}(b) - \xi^{N-1}(b)] \\
+ \int_{(b)}^{(\beta(v))} [(v_1 - \beta^{-1}(u)) \cdot (1 - F^{ex}(v_1)) + \int_{\beta^{-1}(u)}^{v_1} (x - \beta^{-1}(u)) dF^{ex}(x) + \Pi_1(v_2[u]) \cdot dG^{N-1}(u). \right) \] (48)

Note that (48) consists in a sum of terms that are all positive. The equality \( U(v, \beta(v)) = U(v, b) \) would imply the following three equalities:

\[ \left[ \int_{b}^{(\beta(v))} (v_1 - \beta^{-1}(u)) \cdot dG^{N-1}(u) \right] \cdot (1 - F^{ex}(v_1)) = 0; \] (49)
\[ \int_{b}^{(\beta(v))} \left[ \int_{\beta^{-1}(u)}^{v_1} (x - \beta^{-1}(u)) dF^{ex}(x) \right] \cdot dG^{N-1}(u) = 0; \] (50)
\[ \int_{b}^{(\beta(v))} \Pi_1(v_2[u]) \cdot dG^{N-1}(u) = 0. \] (51)

Since \( b < \beta(v) \) and \( v_1 - \beta^{-1}(u) > 0 \) for any \( u \in [b, \beta(v)] \), (49) implies that \( F^{ex}(v_1) = 1 \) so that \( b^{ex} \cup v_1 = b^{ex} \). We have thus either \( b < \hat{\beta}(v_2) \) or \( b < \hat{\beta}(b^{ex}) \). Consider first the case \( b < \hat{\beta}(b^{ex}) \), i.e. \( F^{ex}(\beta^{-1}(b)) < 1 \). We can find thus \( u > b \) in the neighborhood of \( b \) such that \( F^{ex}(\beta^{-1}(u)) < 1 \) and \( u \in S^N_A \). From (50), \( u \in S^N_A \) implies that \( \int_{\beta^{-1}(u)}^{v_1} (x - \beta^{-1}(u)) dF^{ex}(x) = 0 \) and then \( F^{ex}(\beta^{-1}(u)) = F^{ex}(v_1) < 1 \) which raises a contradiction with \( F^{ex}(v_1) = 1 \). Consider now the case \( \hat{\beta}(b^{ex}) \leq b < \hat{\beta}(v_2) \). Take \( x \in
At the ex ante stage, there is positive probability, denoted by \( \hat{p} \), that all the opponents of a given buyer have valuations in the set \( L^*((\hat{x}, \hat{x})) \cap M^*((\hat{\beta}^{-1}(b), \hat{\beta}^{-1}(b))) \) \( \in \Delta(T) \) and then bid in the interval \((b, \hat{\beta}(\hat{x}))\). Conditional on the given bidder's highest opponent bid being in \((b, \hat{\beta}(\hat{x}))\), then the probability that all his opponents have valuations in the set \( L^*((\hat{x}, \hat{x})) \cap M^*((\hat{\beta}^{-1}(b), \hat{\beta}^{-1}(b))) \) \( \in \Delta(T) \) is thus larger than \( \hat{p} \). Under such an event and if the given bidder with type \( v \) has won the first auction with a bid above \( \hat{\beta}(\hat{x}) \) (and so in particular for \( \beta(v) \)), then his continuation payoff is at least \( v_2 - \hat{x} \). Finally we obtain that
\[
\int_b^{(\beta(v))} \Pi_1(v_2 | u) \cdot dG^{N-1}(u) \geq \int_{(\hat{\beta}(\hat{x}))}^{(\hat{\beta}(\hat{x}))} \Pi_1(v_2 | u) \cdot dG^{N-1}(u) \geq \hat{p} \cdot (v_2 - \hat{x}) > 0
\]
which raises a contradiction with (51). On the whole we have shown that \( U(v, b) < U(v, \beta(v)) \).

2b/ In a second step, we consider the case \( b = -1 \). Since we are in a case where \( v_2 \vee \hat{b} < r \) and \( v_1 > r \), then \( \hat{\beta} \left( v_2 \vee (\hat{b} \wedge v_1) \right) > r \) and we have thus from our previous step 2a/ that \( U(v, r) < U(v, \beta(v)) \). To conclude the proof, it is thus sufficient to show that \( U(v, r) \geq U(v, -1) \) for any \( v \in T_r \). From (5) and (6), we have

\[
U(v, r) - U(v, -1) = \left[ \Pi_2^{(v)}(v_2 | r) \cdot G^{N-1}(r) \right] \cdot \left[ G^{N-1}(r) - G^{N-1}(r) \right] + \left[ U_{th}(v, r) - \Pi_2(v_1 | r) \right] \cdot \left[ G^{N-1}(r) - G^{N-1}(r) \right].
\]

From (52), it is sufficient to check that \( c \geq 0 \). If \( r \notin S^A \), then \( c = 0 \). If \( r \in S^A \), then \( N = 2 \) and we have thus \( U_{th}(v, r) - \Pi_2(v_1 | r) = \frac{1}{2} \cdot \left[ (v_1 - r) - \Pi_2(v_1 | r) \right] \cdot \left[ G^{N-1}(r) - G^{N-1}(r) \right] \) so that \( c \geq 0 \).

**M Proof of Proposition 3.5**

Since \( U(v, b) \leq 0 = U(v, -1) \) for any \( b \in B_r \) if \( v \in T \setminus T_r \) and from the RP property, we are left with the equilibrium constraints for \( v \in T_r \). In order to show “global optimality”, let us show that if the RP, WM and LM properties are satisfied, then for any \( v \in T_r \), the function \( b \rightarrow U(v, b) \) is quasi-concave with \( \beta(v) \) being a mode. Note that: 1) the RP and LM properties guarantee that \( \beta(v) \leq v_1 \) for any \( v \in T_r \) and thus in particular that \( \beta(v) = r \) if \( v_1 = r \); 2) the RP and WM properties guarantee that \( \beta(\bar{v}) = \bar{b} \).

As a preliminary, note that \( b \rightarrow U(v, b) \) is differentiable at any point \( b \in S^{NA} \) (i.e. almost everywhere since \( \beta \) is assumed to be regular) and that \( b \rightarrow U(v, b) \) is continuous everywhere except possibly at \( b \in S^A \). In order to prove that \( b \rightarrow U(v, b) \) is quasi-concave with the mode \( \beta(v) \geq r \), it is then sufficient to show that: 1) for any \( b \in S^{NA} \cup (\bar{b}, \infty) \) (and so for almost any point \( b \) such that \( U(v, .) \) is continuous at \( b \)), \( \frac{\partial U(v, b)}{\partial b}(v, b) \geq 0 \) if \( b \leq \beta(v) \) and \( \frac{\partial U(v, b)}{\partial b}(v, b) \leq 0 \) if \( b \geq \beta(v) \); 2) for any \( b \in S^A \cup \{-1\} \), \( \lim_{b \rightarrow b^-} U(v, b) \leq U(v, b) \leq \lim_{b \rightarrow b^+} U(v, b) \).
\lim_{b \to b^+} U(v, \tilde{b}) \text{ if } b \leq \beta(v) \text{ and } \lim_{b \to b^-} U(v, \tilde{b}) \geq U(v, b) \geq \lim_{b \to b^+} U(v, \tilde{b}) \text{ if } b \geq \beta(v). \footnote{We adopt the convention that } \lim_{b \to r^-} U(v, \tilde{b}) := U(v, -1), \lim_{b \to -1^-} U(v, \tilde{b}) := U(v, -1) \text{ and } \lim_{b \to -1^+} U(v, \tilde{b}) := U(v, r).

In a first step, we take \( b \geq \beta(v) \geq r \). If \( b > \tilde{b} \), we have \( \partial U(v, b) \bigg|_{b = \tilde{b}} = 0 \). If \( b \leq \tilde{b} \), we obtain from the intermediate value theorem that there exists \( v' \in \mathcal{L}(v, (\tilde{v}, \bar{v})) \) such that \( \beta(v') = b \). Note that we have thus \( v'_1 \in [v_1, \bar{v}] \). From Lemma 3.8, we have that \( \Pi_1(v_2|b) = 0 \) (since \( \hat{\beta}(v_2) \leq \beta(v) \leq b \) and \( \Pi_1(v_2|b) = 0 \) (since \( \hat{\beta}(v_2) \leq \beta(v') \)) if \( b \in \mathcal{S}^* \cup \mathcal{S}^\downarrow \) and that \( b \in \mathcal{S}^\downarrow \) implies that \( N = 2 \). Furthermore, combining part 4) of Lemma 3.8 with the LM property, we obtain that \( (v'_1 - \beta(v')) - \Pi_2(v'_1|\beta(v')) = 0 \) if \( \beta(v') = b \in \mathcal{S}^N \cup \mathcal{S}^\downarrow \).

Consider first the case \( b \in \mathcal{S}^\downarrow^+. \) We have then \( U_{\text{tie}}(v, b) - \Pi_2(v_1|b) = (v_1 - b) + \Pi_1(v_2|b) - U_{\text{tie}}(v, b) = \frac{1}{2} \cdot [(v_1 - b) - \Pi_2(v_1|b)] \leq \frac{1}{2} \cdot [(v'_1 - b) - \Pi_2(v'_1|b)] = 0 \) (where the inequality comes from Lemma 2.3 and \( v_1 \leq v'_1 \)). This implies that \( U(v, b) - \lim_{b \to b^+} U(v, \tilde{b}) \geq 0 \) and \( \lim_{b \to b^-} U(v, \tilde{b}) - U(v, b) \geq 0 \). Consider then the case \( b \in \mathcal{S}^N. \) We obtain that

\[
\frac{\partial U(v, b)}{\partial b} = [(v_1 - b) - \Pi_2(v_1|b)] \cdot (N - 1)(G(b))^{N-2}g(b)
\]

where the last inequality comes from Lemma 2.3 and \( v_1 \leq v'_1 \).

In a second step, we take \( b \in [r, \beta(v)] \). We obtain from the intermediate value theorem that there exists \( v'^{'} \in \mathcal{L}(r, v, r) \) such that \( \beta(v'^{'})) = b \). Note that \( v'_1 \in [r, v_1] \). From Lemma 3.8 and the LM property, we obtain that \( (v'_1 - \beta(v'^{'})) - \Pi_2(v'_1|\beta(v'^{'})) = 0 \) if \( b \in \mathcal{S}^N \cup \mathcal{S}^\downarrow \). Consider first the case \( \beta(v') = b \in \mathcal{S}^\downarrow^+ \) (so that \( N = 2 \)). We have then

\[
U_{\text{tie}}(v, b) - \Pi_2(v_1|b) = (v_1 - b) + \Pi_1(v_2|b) - U_{\text{tie}}(v, b) = \frac{1}{2} \cdot [(v_1 - b) + \Pi_1(v_2|b) - \Pi_2(v_1|b)] \geq \frac{1}{2} \cdot [(v'_1 - b) - \Pi_2(v'_1|b)] = 0 \end{equation}

and also since \( \Pi_1(\cdot, \cdot) \geq 0 \). This implies that \( U(v, b) - \lim_{b \to b^+} U(v, \tilde{b}) \leq 0 \) and if \( b > r \), \( \lim_{b \to b^-} U(v, \tilde{b}) - U(v, b) \leq 0 \). Consider then the case \( b \in \mathcal{S}^N. \) We obtain that

\[
\frac{\partial U(v, b)}{\partial b} \geq [(v_1 - b) - \Pi_2(v_1|b)] \cdot (N - 1)(G(b))^{N-2}g(b) = [(v_1 - \Pi_2(v_1|b)) - (v'_1 - \Pi_2(v'_1|b))] \cdot (N - 1)(G(b))^{N-2}g(b) \geq 0 \text{ where the last inequality comes from Lemma 2.3 and } v_1 \geq v'_1.
\]

In order to check that \( b \to U(v, b) \) is quasi-concave and with the mode \( \beta(v) \), it remains to check that \( U(v, r) - U(v, -1) \geq 0 \) if \( v \in \mathfrak{T}_r \) which has been proved independently in case 2b/ of the proof of Proposition 3.4.

## N Proof of Proposition 3.6

### Only if’ Part

First, we have from Lemma 3.7 that \( \hat{\beta} \) is strictly increasing and \( \mathcal{S} = [\hat{\beta}(r), \hat{\beta}(\bar{v})] \) which implies that \( \bigcup_{x \in [r, \bar{v}]} I_\beta(x) = \mathcal{P} \). From the RP property, we ob-
tain finally that \( \bigcup_{x \in [r, \bar{v}]} I_B(x) = T_r \). Second, the RP and WM properties (which hold for any equilibrium) imply that for any \( x \in [r, \bar{v}] \), we have \( I_B(x) \subseteq T_r \setminus \{L^*((x, x)) \cup M^*((x, x))\} \).

Furthermore, from Lemma 3.9, we obtain that \( I_B(x) = \{v \in T : v_1 = x\} \) if \( N \geq 3 \) or if \( N = 2 \) and \( x < b^{ex} \). We conclude this part after noting that the LM property (which holds for any equilibrium) implies that (16) holds for \( v \in D \) which is equivalent to (20).

‘If’ Part From Proposition 3.5, it is sufficient to check that the RP, WM and LM properties hold. The RP has been directly assumed. Let us check the WM property. Take \( v, v' \in T_r \). Since \( \bigcup_{x \in [r, \bar{v}]} I_B(x) = T_r \), then there exist \( x, x' \in [r, \bar{v}] \) such that \( v \in I_B(x) \) and \( v' \in I_B(x') \). From (19), we have \( x \in [v_2, v_1] \) and \( x' \in [v'_2, v'_1] \). If \( v' \in L^*(v) \) [resp. \( M^*(v) \)], then \( x' < x \) [resp. \( x' > x \)]. Since \( \hat{\beta} \) is strictly increasing, this implies thus that \( \hat{\beta}(x) > \hat{\beta}(x') \) [resp. \( \hat{\beta}(x) < \hat{\beta}(x') \)] and we are done with the first part of the WM property. The second part comes directly from (19) and \( \hat{\beta} \) strictly increasing. Let us now check the LM property. Take \( v \in T_r \) such that \( \beta(v) \in S^{NA}_2 \cup S^A_2 \). Let \( x \in [r, \bar{v}] \) such that \( v \in I_B(x) \). On the one hand, if \( N \geq 3 \) or \( N = 2 \) and \( x < b^{ex} \), then \( v_1 = x \) so that (20) implies (16). On the other hand, if \( N = 2 \) and \( x \geq b^{ex} \), then the RP and WM properties imply that \( x \leq v_1 \) and \( H_2(u|\beta(v)) = F^{ex}(u) = 1 \) for \( u \geq x \) (see Lemma 3.8). On the whole (20) implies (16) which concludes the proof.

\( \Box \) Proof of Lemma 3.10

Since we have assumed that \( F_1(\cdot) \) is continuously differentiable on \([r, \bar{v}]\) while \( F_1(x) > 0 \) if \( x > r \) and \( F_2(x) \) is continuously differentiable on \([x \lor r, \bar{v}]\) for any \( x \in [0, \bar{v}] \), then we obtain that \( \psi \) is continuously differentiable on \([r, \bar{v}]\). Furthermore, we have \( \psi'(x) = 1 - F^{ex}(x) + \int_r^x (N - 2) \frac{dF_1'(u)}{F_1'(x)} \left[ \frac{F_1(u)}{F_1(x)} \right]^{N-2} F_2(u|x) F^{ex}(u) du - \int_r^x \frac{F_1(u)}{F_1(x)} \frac{dF_2(u|x)}{dx} F^{ex}(u) du \) for any \( x \in (r, \bar{v}] \). A2 implies that \( \frac{dF_2(u|x)}{dx} \leq 0 \) for any \((x, u) \in T_r\) (with \( u < \bar{v} \)). If \( r < x < b^{ex} \), we have \( 1 - F^{ex}(x) > 0 \) and we obtain then that \( \psi'(x) > 0 \). If \( x \geq b^{ex} \) and \( x > r \), we have then \( \frac{dF_1(u)}{F_1(x)} = \int_0^r f(x, v_2) dv_2 > 0 \), \( F_2(x|x) = 1 > 0 \) and \( F^{ex}(x) = 1 > 0 \). As a corollary, we obtain that \( \int_r^x (N - 2) \frac{dF_1'(u)}{F_1'(x)} \left[ \frac{F_1(u)}{F_1(x)} \right]^{N-2} F_2(u|x) F^{ex}(u) du > 0 \) and thus \( \psi'(x) > 0 \).

Remark O.1 If \( F_1(r) > 0 \) and \( b^{ex} = r \), then we obtain also from the analysis above that \( \psi \) is differentiable at \( r \) with \( \psi'(r) = 0 \) (independently of A2). This feature helps to understand why the R1-equilibrium does not extend in some possible variations of the model. E.g. consider the case with no extra bidder but where the two units are not exactly identical but where buyers’ value [resp. the reserve price] for the second unit for sale is a fixed share \( (1 - \alpha) \geq 0 \) of their value [resp. the reserve price] for the first unit, as already discussed in Remark 2.1. For an equilibrium bid function \( \beta \) that depends solely
on the first valuation and that is strictly increasing in it, the first-order condition yields
\[ \beta(v) = v_1 - (1 - \alpha) \cdot \int_r^v \frac{F_1(u)}{F_1(v_1)} N^{-2} F_2(u|v_1) du := \psi(v_1) \] for any \( v \in T_r \). If \( \alpha \geq 0 \), this extension fits actually in our model and \( \beta \) corresponds to the R1-equilibrium. On the contrary, if \( \alpha < 0 \), then \( F_1(r) > 0 \) would imply \( \psi'(r) < 0 \) which would raise a contradiction. On the whole, equilibria should be inefficient when \( N \geq 3 \) and \( F_1(r) > 0 \).

\[ P \quad \text{Proof of Proposition 3.7} \]

**Preliminary remark** From A1, we obtain that \( \psi \) is strictly increasing. For the R1-equilibrium, this means in particular that \( \bar{b} = \psi(\bar{v}) \) so that \( \psi^{-1} \) is defined from \( S \) to \( [r,\bar{v}] \). Next we also let \( \psi^{-1}(b) = v \) for \( b > \bar{b} \). Since \( \psi \) is continuously differentiable on \( (r,\bar{v}] \) with \( \psi' > 0 \), we obtain that \( \psi^{-1} \) is continuously differentiable on \( (r,\psi(\bar{v})) \) with
\[ (\psi^{-1})'(b) = \frac{1}{\psi'(\psi^{-1}(b))} > 0. \]

**R1-equilibrium** We first check that \( \beta_{R1} \) is a regular bid function. 1) \( \mathcal{P} = T_r \) is a closed set. 2) \( \beta_{R1}(v) = \psi(v_1) \) for any \( v \in T_r \) so that \( \beta_{R1} \) is continuous on \( T_r \) and thus on \( \mathcal{P} \). We have also \( G(b) = F_1(\psi^{-1}(b)) \) if \( b \geq r \). Combined with our preliminary remark, we obtain thus 3) \( G(.) \) has a finite number of atoms (the unique possible atom is at \(-1 \) once \( r > 0 \)); 4) \( G(.) \) is continuously differentiable almost everywhere (in particular at any bid \( b \in (r,\bar{b}) \) where we have \( g(b) = \frac{dF_1}{dx}(\psi^{-1}(b)) \cdot \frac{1}{\psi'(\psi^{-1}(b))} > 0 \). For any \( b \geq r \), we have \( \{ v \in T : \beta_{R1}(v) \leq b \} = \{ v \in T : v_1 \leq \psi^{-1}(b) \} \) and then
\[ F_1(x|\beta_{R1} \leq b) = \int_0^x f(v) dv = \frac{F_1(x\wedge\psi^{-1}(b))}{F_1(\psi^{-1}(b))}. \]
Then at any \( b \in (r,\bar{b}) \) (which guarantees that \( \psi^{-1} \) is continuously differentiable at \( b \)) and thus for any \( b \in S^{NA} \), we have that the function
\[ u \rightarrow F_1(x|\beta_{R1} \leq u) \]
is continuously differentiable at \( b \) for any \( x \) except possibly for \( x = \psi^{-1}(b) \). For any \( b \geq r \), we have \( F_2(x|\beta_{R1} \leq b) = \int_0^\psi^{-1}(b) f(v) dv \). If we have also \( b \in (r,\bar{b}) \), the numerator is continuously differentiable as a function of \( b \) (the derivative is given by \( (\psi^{-1})'(b) \cdot \int_0^\psi f(\psi^{-1}(b),v_2) dv_2 \), while the denominator is continuously differentiable as a function of \( b \) and the derivative is strictly positive (the derivative is given by \( (\psi^{-1})'(b) \cdot F_1(\psi^{-1}(b)) \)). The function \( u \rightarrow F_2(x|\beta_{R1} \leq u) \) is thus continuously differentiable on \( S^{NA} \) for any \( x \). On the whole, part 5) of the definition of a regular bid function has been verified.

We can then apply Proposition 3.6. What remains to check for \( \beta_{R1} \) is the local martingale property (20) which is a corollary of the expression (21) after an integration by parts and having noted that
\[ H_2(u|\psi(x)) = \left[ \frac{F_1(u)}{F_1(x)} \right]^{N-2} \cdot F_2(u|x) \cdot F^{ex}(u) \text{ if } x \in (r,v). \]

**R2-equilibrium** We first check that \( \beta_{R2} \) is a regular bid function. 1) \( \mathcal{P} = T_r \) is a closed set. 2) \( \beta_{R2} \) is continuous on \( T_r \) and thus on \( \mathcal{P} \). We have also \( G(b) = F_1(b) \) [resp. \( G(b) = F_2(b) \) if \( b \in [r,r') \) [resp. \( b \in [r',\bar{b}] \). Then: 3) \( G(.) \) has a finite number of atoms (the unique possible atoms are at \(-1 \) once \( r > 0 \) and \( r' \) once \( r' > 0 \)); 4) \( G(.) \) is continuously differentiable almost everywhere (in particular at any bid \( b \in (r,\bar{b}) \) where we have \( g(b) = \frac{dF_1}{dx}(\psi^{-1}(b)) \cdot \frac{1}{\psi'(\psi^{-1}(b))} > 0 \). For any \( b \geq r \), we have \( \{ v \in T : \beta_{R2}(v) \leq b \} = \{ v \in T : v_1 \leq \psi^{-1}(b) \} \) and then
\[ F_1(x|\beta_{R2} \leq b) = \int_0^\psi^{-1}(b) f(v) dv = \frac{F_1(x\wedge\psi^{-1}(b))}{F_1(\psi^{-1}(b))}. \]
Then at any \( b \in (r,\bar{b}) \) (which guarantees that \( \psi^{-1} \) is continuously differentiable at \( b \)) and thus for any \( b \in S^{NA} \), we have that the function
\[ u \rightarrow F_1(x|\beta_{R2} \leq u) \]
is continuously differentiable at \( b \) for any \( x \) except possibly for \( x = \psi^{-1}(b) \). For any \( b \geq r \), we have \( F_2(x|\beta_{R2} \leq b) = \int_0^\psi^{-1}(b) f(v) dv \). If we have also \( b \in (r,\bar{b}) \), the numerator is continuously differentiable as a function of \( b \) (the derivative is given by \( (\psi^{-1})'(b) \cdot \int_0^\psi f(\psi^{-1}(b),v_2) dv_2 \), while the denominator is continuously differentiable as a function of \( b \) and the derivative is strictly positive (the derivative is given by \( (\psi^{-1})'(b) \cdot F_1(\psi^{-1}(b)) \)). The function \( u \rightarrow F_2(x|\beta_{R2} \leq u) \) is thus continuously differentiable on \( S^{NA} \) for any \( x \). On the whole, part 5) of the definition of a regular bid function has been verified.

We can then apply Proposition 3.6. What remains to check for \( \beta_{R1} \) is the local martingale property (20) which is a corollary of the expression (21) after an integration by parts and having noted that
\[ H_2(u|\psi(x)) = \left[ \frac{F_1(u)}{F_1(x)} \right]^{N-2} \cdot F_2(u|x) \cdot F^{ex}(u) \text{ if } x \in (r,v). \]
differentiable almost everywhere (in particular at any bid \(b \in (r, r') \cup (r', b_0)\)). More precisely, we have \(S^{NA} \subseteq (r, r') \cup (r', b_0)\). Consider then \(b \in S^{NA}\). If \(b \in (r, r')\) [resp. \(b \in (r', b_0)\)], we have \(\{v \in T : \beta_R(v) \leq b\} = \{v \in T : v_1 \leq b\}\) [resp. \(\{v \in T : \beta_R(v) \leq b\} = \{v \in T : v_2 \leq b\}\)]. We have then: A) \(F_1(x|\beta_R \leq b) = \frac{F_1(x|b)}{F_1(b)}\) and \(F_2(x|\beta_R \leq b) = \frac{F_2(x|b)}{F_2(b)}\) if \(b \in (r, r')\); B) \(F_1(x|\beta_R \leq b) = \frac{\int_{b}^{\infty} f(v)dv}{F_1(b)}\) and \(F_2(x|\beta_R \leq b) = \frac{\int_{b}^{\infty} f(v)dv}{F_2(b)}\) if \(b \in (r', b_0)\).

We obtain on the whole that the function \(u \to F_i(x|\beta_R \leq u)\) (\(i = 1, 2\)) is continuously differentiable on \(S^{NA}\) for any \(x\) except possibly for \(x = b\) and part 5 of the definition of a regular bid function is thus satisfied.

Let us now check that \(\beta_{R_2}\) is an ex-post equilibrium which implements the Vickrey payoffs. Consider a given buyer with type \(v \in T\) and that the type of his opponent is \(v^* \in T\) so that he bids \(b^* := \beta_R(v^*)\). We consider below four exhaustive cases for the realizations of \(b^*\). First case: \(b^* > r'\) so that \(v_2^* = b^*\). If \(v_1 \geq b^* \geq v_2\), then for any bid (and so in particular for the bid \(b = \beta_R(v)\)), the given buyer payoff equals \(v_1 - v_2^*\) which corresponds to his Vickrey payoff. On the contrary, if \(v_1 < b^*\), then he receives a strictly negative payoff when bidding above \(b^*\) while his payoff is null otherwise (and so in particular for the bid \(b = \beta_R(v) \leq v_1 < b^*\)) which corresponds to his Vickrey payoff. Finally, if \(v_2 > b^*\), then his payoff equals \((v_1 - v_2^*) + (v_2 - v_1^*) \lor 0\) (which corresponds to his Vickrey payoff) if he bids strictly above \(b^*\) (so in particular for the bid \(b = \beta_R(v) = v_2 > b^*\)) while his payoff equals \(v_1 - v_2^*\) [resp. \((v_1 - v_2^*) + \frac{1}{2} \cdot (v_2 - v_1^*) \lor 0\)] if he bids strictly below \(b^*\) [resp. if he bids exactly \(b^*\)]. \(\beta_{R_2}(v)\) is thus the best reply for any realization of \(v\).

Second case: \(b^* = r'\) so that \(v_1^* \geq r' \geq v_2^*\). If \(v_1 \geq r' \geq v_2\), then for any bid (and so in particular for \(b = \beta_{R_2}(v)\)), the given buyer payoff equals \(v_1 - r'\) which corresponds to his Vickrey payoff (because \(r'\) corresponds to the valuation of the extra bidder). On the contrary, if \(v_1 < b^*\), then he receives a strictly negative payoff when bidding above \(b\) while his payoff is null otherwise (and so in particular for the bid \(b = \beta_{R_2}(v) \leq v_1 < b^*\)) which corresponds to his Vickrey payoff. Finally, if \(v_2 > b^*\), then his payoff equals \((v_1 - r') + (v_2 - v_1^*) \lor 0\) (which corresponds to his Vickrey payoff) if he bids strictly above \(b^*\) (so in particular for the bid \(b = \beta_{R_2}(v) = v_2 > b^*\)) while his payoff equals \(v_1 - r'\) [resp. \((v_1 - r') + \frac{1}{2} \cdot (v_2 - v_1^*) \lor 0\)] if he bids strictly below \(b^*\) [resp. if he bids exactly \(b^*\)]. \(\beta_{R_2}(v)\) is thus the best reply for any realization of \(v\).

Third case: \(b^* = -1\) so that \(v_1^* < r\). If \(v_1 < r\), it is straightforward, that the given buyer’s best reply is not to participate and to receive then the null payoff which corresponds to his Vickrey payoff. If \(v \in T_r\), submitting an active bid (e.g. the bid \(b = \beta_{R_2}(v) \geq r\)) is a best reply which leads to the payoff \((v_1 - r) + (v_2 - r') \lor 0\) which corresponds to his Vickrey payoff (because \(r'\) corresponds to the valuation of the extra bidder). Fourth case: \(r \leq b^* < r'\) so that \(v_1^* = b^*\). If \(v_1 < b^*\), then the given buyer receives a strictly negative
payoff when bidding above $b^*$ while his payoff is null otherwise (and so in particular for the bid $b = \beta R_2(v) \leq v_1 < b^*$) which corresponds to his Vickrey payoff. If $v_1 > b^*$, then his payoff equals $(v_1 - v_1^1) + (v_2 - r') \lor 0$ (which corresponds to his Vickrey payoff) if he bids strictly above $b^*$ (so in particular for the bid $b = \beta R_2(v) \geq v_2 > b^*$) while his payoff equals $(v_1 - r') \lor 0$ [resp. $\frac{1}{2} \cdot [(v_1 - r') \lor 0] + \frac{1}{2} \cdot [(v_1 - v_1^1) + (v_2 - r')] \lor 0$] if he bids strictly below $b^*$ [resp. if he bids exactly $b^*$]. $\beta R_2(v)$ is thus the best reply for any realization of $v$.

On the whole, we have shown in the R2-equilibrium that for any realization in $T$ of the type of the given buyer’s opponent, then $\beta R_2(v)$ is a best reply for any given buyer with type $v \in T$ and that it yields the Vickrey payoff.

Q Proof of Proposition 3.9

In the R1-equilibrium, we have $F_2(u|b) = F_2(u|\tilde{\beta}^{-1}(b))$ for any $b \in S$ (see the proof of Lemma 3.9). In order to apply Proposition 3.3, we have then to check that $F_2(u|x) \leq F_2(u|x')$ for any $x, x' \in [r, \overline{r}]$ with $x \geq x'$. In other words, we have to check that A2 holds which has been assumed directly.

Consider now the R2-equilibrium. In particular, this means that $N = 2$. To show that prices are nondecreasing, it is sufficient to consider the events where $p^1 > r$. Consider first the cases where $p^1 \leq r'$ so that $v_1^{(2:N)} \geq p^1$. In the second stage we have $p^2 \geq v_1^{(2:N)}$ and thus $p^2 \geq p^1$. Consider then the cases where $p^1 > r'$ so that $p^1 = v_2^{(2:N)} \leq v_2^{(1:N)}$. There are two subcases: 1) the winner of the first stage wins the second unit and $p^2 \geq v_2^{(2:N)} \geq v_2^{(2:N)} = p^1$; 2) the winner of the first stage loses the second unit and $p^2 \geq v_2^{(1:N)} \geq v_2^{(2:N)} = p^1$. In any case, we have $p^2 \geq p^1$.

R Proof of Proposition 3.10

In the R1-equilibrium, we have $\beta R_1(v) = \tilde{\beta} R_1(v_1)$ for all $v \in T$. From Proposition 3.4, we obtain then that $\beta R_1(v) = \max \{b \in S_U^{\text{max}}(v)\}$ for almost all $v \in \mathcal{P}$. Conversely, consider an equilibrium bid function $\beta$ such that $\beta(v) = \max \{b \in S_U^{\text{max}}(v)\}$ for almost all $v \in \mathcal{P}$. From Proposition 3.4 and by continuity, this means that $\beta(v) = \tilde{\beta}(v_1)$ for all $v \in \mathcal{P}$. Thanks to Remark 3.2, we conclude that $\beta = \beta R_1$.

In the R2-equilibrium, we have $\beta R_2(v) = v_2 = \tilde{\beta} R_2(v)$ if $v_2 > r'$, $\beta R_2(v) = r' = \tilde{\beta} R_2(r')$ if $v_2 \leq r' \leq v_1$ and $\beta R_2(v) = v_1 = \tilde{\beta} R_2(v_1)$ if $r \leq v_1 < r'$. On the whole we obtain that $\beta R_2(v) = \tilde{\beta} R_2(v_2 \lor (b^{ex} \land v_1))$ for all $v \in \mathcal{P}$. From Proposition 3.4, we obtain then that $\beta R_2(v) = \min \{b \in S_U^{\text{max}}(v) \\setminus \{-1\}\}$ for almost all $v \in \mathcal{P}$. Conversely, consider an equilibrium bid function $\beta$ such that $\beta(v) = \min \{b \in S_U^{\text{max}}(v) \\setminus \{-1\}\}$ for almost all $v \in \mathcal{P}$. From Proposition 3.4 and by continuity, this means that $\beta(v) = \tilde{\beta}(v_2 \lor (b^{ex} \land v_1))$ for all
$v \in \mathcal{P}$. If $b_{ex}^v = \bar{v}$, for any equilibrium, we have then $\beta(v) = v_1 = \beta_{R_1}(v)$ for all $v \in \mathcal{T}_r$ and so $\beta = \beta_{R_1}$. If $b_{ex}^v < \bar{v}$, we conclude the proof thanks to Remark 3.7.

S Complement to Subsection 3.3.3 and proof of Proposition 3.11

Consider a given equilibrium candidate $\beta_\alpha$ with $\alpha \in (0, 1)$. For any $x \in (r, \bar{v})$, we have then

$$G(\hat{\beta}_\alpha(x)) = \int_0^x \left[ \int_0^{\hat{\beta}_\alpha(x)} f(v)dv \right] dv_1 = G(r) + \int_r^x \frac{1}{1 - \alpha} \cdot \int_s^{\hat{\beta}_\alpha(x)} f(u, s - au)du \, ds,$$

$$\frac{dG(\hat{\beta}_\alpha(x))}{dx} = \frac{1}{1 - \alpha} \cdot \int_x^{\hat{\beta}_\alpha(x)} f(u, x - au)du,$$

$$F_2(s|\beta_\alpha \leq \hat{\beta}_\alpha(x)) = \frac{\int_0^x \left[ \int_0^{\hat{\beta}_\alpha(x)} f(v)dv \right] dv_1}{\int_0^x f(v)dv}, \quad \text{for all} \quad x \in (r, \bar{v})$$

First case: $x \leq \alpha \bar{v}$. By means of a change of variable, we obtain

$$F_2(s|\hat{\beta}_\alpha(x)) = \frac{\int_{[1 - \frac{1}{\alpha}]s}^{1} f(x + t \frac{1 - \alpha}{\alpha} x, x - tx)dt}{\int_0^1 f(x + t \frac{1 - \alpha}{\alpha} x, x - tx)dt}. \quad (53)$$

Second case: $x \geq \alpha \bar{v}$. By means of a change of variable, we obtain

$$F_2(s|\hat{\beta}_\alpha(x)) = \frac{\int_{[1 - \frac{1}{\alpha}]s}^{1} f(x + t \cdot (\bar{v} - x), x - \frac{1}{1 - \alpha} \cdot t \cdot (\bar{v} - x))dt}{\int_0^1 f(x + t \cdot (\bar{v} - x), x - \frac{1}{1 - \alpha} \cdot t \cdot (\bar{v} - x))dt}. \quad (54)$$

We have then an explicit formula for $\hat{\beta}_\alpha(x) = x - \int_r^x F_2(s|\hat{\beta}_\alpha(x))ds$.

For the uniform distribution, (53) and (54) reduce to $F_2(s|\hat{\beta}_\alpha(x)) = \frac{x}{\bar{v}} \wedge 1$ if $x \leq \alpha \bar{v}$ and $F_2(s|\hat{\beta}_\alpha(x)) = \left[ \frac{\alpha + (1 - \alpha)(x - \bar{v})}{\alpha(1 - \alpha)} \right] \wedge 0$ if $x \geq \alpha \bar{v}$. We obtain then that $\hat{\beta}_\alpha(x) = \frac{x^2 + x^2}{2x}$ if $x \leq \alpha \bar{v}$, $\hat{\beta}_\alpha(x) = r + \frac{(1 - \alpha)(x - r)^2}{2\alpha(\bar{v} - x)}$ if $\alpha \bar{v} + (1 - \alpha)r \geq x \geq \alpha \bar{v}$ and $\hat{\beta}_\alpha(x) = \frac{x}{\bar{v}} + \frac{(x - \alpha \bar{v})}{2(\bar{v} - x)}$ if $x \geq \alpha \bar{v} + (1 - \alpha)r$. $\hat{\beta}_\alpha$ is then differentiable almost everywhere on $[r, \bar{v}]$ and we have $\hat{\beta}_\alpha'(x) = \frac{1}{2}(1 - \frac{r^2}{x^2}) \geq 0$ if $x > \alpha \bar{v}$, $\hat{\beta}_\alpha'(x) = \frac{(1 - \alpha)(x - r)}{\alpha(\bar{v} - x)} + \frac{(1 - \alpha)(x - r)^2}{2\alpha(\bar{v} - x)^2} \geq 0$ if $\alpha \bar{v} + (1 - \alpha)r > x > \alpha \bar{v}$ and $\hat{\beta}_\alpha'(x) = \frac{1 - \alpha}{2(1 - \alpha)} > 0$ if $x > \alpha \bar{v} + (1 - \alpha)r$. On the whole, we obtain that $\hat{\beta}_\alpha$ is strictly increasing on $[r, \bar{v}]$ for any $\alpha \in (0, 1)$ which allows us to conclude that $\beta_\alpha$ is an equilibrium. For $\alpha = 1$, we have $\psi(x) = \frac{x^2 + x^2}{2x}$ for any $x \in [r, \bar{v}]$ so that A1 holds.

Proof of Proposition 3.11 We show that for any given distribution $F$ and $r > 0$, then $\hat{\beta}_\alpha$ is strictly increasing on $[r, \bar{v}]$ when $\alpha$ is sufficiently close to 0. For this, it is sufficient to check that $\frac{dF_2(s|\hat{\beta}_\alpha(x))}{ds} < 0$ uniformly on $s \in \left( \left[ \frac{x - \alpha \bar{v}}{1 - \alpha} \right] \wedge r, x \right)$ and $x \in [r, \bar{v}]$ when $\alpha$ is
sufficiently close to 0.

If $\alpha < \frac{1}{2}$, then we obtain from (54) that

$$\frac{dF_2(s|\tilde{\beta}_\alpha(x))}{dx} = -\frac{1}{\alpha} \left[ \frac{1}{u - x} + \frac{x - s}{(u - x)^2} \right] \int_0^1 f(x + \frac{1}{\alpha} (x - s), s) f(x + t \cdot (v - x), x - \frac{1}{\alpha} t \cdot (v - x)) dt + O(1)$$

for any $x \in [r, \overline{v}]$ and $s \in \left( \frac{1}{r - \overline{v}} \right) \cap r, x$ where $O(1)$ is uniformly bounded as a function of $\alpha$, $s$ and $x$ (it is where we use our additional assumption on the continuous differentiability of $f$ on the compact set $T_r$ such that that the derivatives $\frac{\partial f(v)}{\partial v_1}$ and $\frac{\partial f(v)}{\partial v_2}$ are uniformly bounded). On the whole, there exists a threshold $\alpha_0$ such that $\frac{dF_2(s|\tilde{\beta}_\alpha(x))}{dx} < 0$ for any $x \in [r, \overline{v}]$ and $s \in \left( \frac{1}{r - \overline{v}} \right) \cap r, x$ once $\alpha \leq \alpha_0$. Q.E.D.

### T Proof of Proposition 4.1

We are left with the case $N = 2$ as it is assumed from now on. As with risk-neutral buyers, since $N = 2$, we have that (27a) and (27b) hold also for any $v \in T_r$ such that $\beta(v) \in S^A_r$. Consequently, (27a) holds for almost all $v \in T_r$, which implies that $\beta(v) \leq v_1$. By continuity and from the RP property (which results from efficiency), we have thus $\beta(v) \leq v_1$ for any $v \in T$, which further implies that $\overline{b} = r$. Efficiency guarantees the WM property, which implies that $\beta(\overline{v}) > \beta(v)$ if $\overline{v} \in M^*(v)$, which implies that $F_2(v_1|I(\beta(v), \epsilon)) = 1$ for any $\epsilon < 0$ [resp. $\epsilon = 0$] and $v \in T_r$ with $\beta(v) > r$ [resp. $\beta(v) \in S^A_r$]. On the whole, we obtain that $F_2(u|\beta(v)) = 1$ if $u \geq v_1$ and $\beta(v) \in S^{NA} \cup S^A_r$. Efficiency implies also that on the set $T_r \cap L((x^*, x^*))$, the bidding function $\beta$ is strictly increasing in $v_1$, does not depend on $v_2$ so that $\beta(v) = \hat{\beta}(v_1)$. Efficiency implies also that $\beta(v) \geq \hat{\beta}(x^*)$ for any $v \in T_r \setminus L((x^*, x^*))$ and also that $\hat{\beta}(v) = \overline{b} > \hat{\beta}(x^*)$ (note that $x^* < \overline{v}$ here). This further implies that

$$H_2(u|\beta(v)) = F_2(u|\beta(v)) \cdot F^{ex}(u) = F_2(u|v_1) \cdot F^{ex}(u)$$

for any $v \in T_r \cap L^*((x^*, x^*))$. From (27a) and by continuity, we have thus

$$\beta(v) = v_1 - \mathcal{V}^{-1}(\int_{u,v} \mathcal{V}(v_1 - u)d[F_2[u|v_1] \cdot F^{ex}(u)])$$

for any $v \in T_r \cap L((x^*, x^*))$.

With the same argument as the one invoked for $N \geq 3$, we have that the stochastic variable $[v_1 - u] \vee 0$ where $u$ is distributed according to $H_2(.|\beta(v))$ is not a random variable for any $v \in T_r$ such that $\beta(v) \in S^{NA} \cup S^A_r$ and $v_1 \geq x^*$. From (27a), we obtain then that for any $v \in T_r$ such that $\beta(v) \in S^{NA} \cup S^A_r$ and $v_1 \geq x^*$, we have: either $1/ \beta(v) = v_1$
and $H_2(u|\beta(v)) = 0$ if $u < v_1$; or 2/ $\beta(v) < v_1$ and $H_2(u|\beta(v)) = 1[u \geq \beta(v)]$ for any $u \in \mathbb{R}$. Since we have shown above that $F_2(u|\beta(v)) = 1$ if $u \geq v_1$, we obtain in case 1/ that $H_2(u|\beta(v)) = 1[u \geq v_1]$ for any $u \in \mathbb{R}$. On the whole, we have $H_2(u|b) = 1[u \geq b]$ for any $b \in S^N \cup S^I_+$ and thus $F_2(u|b) = 1[u \geq b]$ if we have also $b > \hat{\beta}(x^*)$.

We now show that for any $b \in (\hat{\beta}(x^*), \overline{b}]$ and $v \in T$, then $\beta(v) = b$ implies $v_2 = b$. Suppose on the contrary that there exists $b \in (\hat{\beta}(x^*), \overline{b}]$ and $v \in T \setminus L((x^*, x^*))$ such that $\beta(v) = b$ and $v_2 \neq b$. Pick then $\epsilon > 0$ small enough such that $v_2 \notin [b - \epsilon, b + \epsilon]$ and $b - \epsilon > \hat{\beta}(x^*)$. Since $\beta$ is continuous, there is $\alpha > 0$ small enough such that $[v_2 - \alpha, v_2 + \alpha] \cap [b - \epsilon, b + \epsilon] = \emptyset$ and $Z(v, \alpha) \subseteq I(b, \epsilon) \cup I(b, -\epsilon)$. Since $Z(v, \alpha) \in \Delta(T)$, we obtain

$$F_2(v_2 + \alpha|I(b, \epsilon) \cup I(b, -\epsilon)) - F_2(v_2 - \alpha|I(b, \epsilon) \cup I(b, -\epsilon)) \geq \frac{\text{Prob}(Z(v, \alpha))}{\text{Prob}(I(b, \epsilon) \cup I(b, -\epsilon))} > 0.$$ (57)

However, we have also $F_2(u|I(b, \epsilon) \cup I(b, -\epsilon)) = \int_{b - \epsilon}^{b + \epsilon} F_2(u|x)dG(x) = \int_{b - \epsilon}^{b + \epsilon} 1[u \geq x]dG(x)$ which raises a contradiction with (57) since $[v_2 - \alpha, v_2 + \alpha] \cap [b - \epsilon, b + \epsilon] = \emptyset$.

Finally, for any $v \in T \setminus L((x^*, x^*))$, we have either $\beta(v) = \hat{\beta}(x^*)$ or $\beta(v) = v_2 > \hat{\beta}(x^*)$. If $v \in M^*((x^*, x^*))$, the WM property guarantees that $\beta(v) > \hat{\beta}(x^*)$ and thus that $\beta(v) = v_2$. By continuity, we have thus $\beta(v) = v_2$ if $v \in M((x^*, x^*))$ and in particular $\beta(x^*) = x^*$. If $v \in T \setminus \{L((x^*, x^*)) \cup M((x^*, x^*))\}$, we have then $v_2 < x^* = \hat{\beta}(x^*)$ which implies thus that $\beta(v) = \hat{\beta}(x^*) = x^*$.

From (56), $\hat{\beta}(x^*) = x^*$ implies that $\int_{x^*}^{x^*} V(x^* - u)d[F_2^x[u|x^*] \cdot F^{ex}(u)] = 0$. If $\frac{b}{2} < x^*$, then take $x = \frac{b^x + x^*}{2}$. We have then $\int_{x^*}^{x^*} V(x^* - u)d[F_2^x[u|x^*] \cdot F^{ex}(u)] \geq V(x^* - x) \cdot F_2^x[x|x^*] \cdot F^{ex}(x) > 0$ (in particular since $x > \frac{b}{2}$) which raises a contradiction. We have thus shown that there does not exist an efficient equilibrium if $\frac{b}{2} < x^*$. We consider now the case where $\frac{b}{2} = x^*$. If we gather the necessary conditions we have shown above, we have precisely $\beta = \beta_{R2}$, i.e. efficient equilibria correspond to the R2-equilibrium under risk aversion.

### U Proof of Proposition 4.2

In a first step we show that the function $\psi^V$, where $\psi^V(x) := \beta^V_{R1}((x, x))$ for any $x \in [r, \overline{r}]$, is continuously differentiable on $(r, \overline{r}]$ with $(\psi^V)' > 0$. Note that $\beta^V_{R1}(v) = \psi^V(v_1)$ for any $v \in T_r$. With an integration per part, we have equivalently $\psi^V(x) = x - V^{-1}\left(\int_{r}^{x} \frac{f_{1}(x)}{f_{1}(x)}^{N-2} F_2(u|x) F^{ex}(u) d[-V(x - u)]\right)$. With the same arguments as in the proof of Lemma 3.10, we obtain that $\psi^V$ is continuously differentiable on $(r, \overline{r}]$ with
it is straightforward as with risk-neutral buyers. Consider now for any \( \rho \neq 0 \) implies that \( \varphi(\rho) \geq \varphi'(\rho) \rho \) for all stochastic variable \( Y \). In particular, this means that the right-hand side of (58) is positive and thus that \( \psi^V(x) > 0 \).

If \( x \geq b^x \) and \( x > r \), we have \( \frac{dF(x)}{dx} = f(x) + v_2 > 0 \), \( F_2(x) = 1 > 0 \) and \( F^x(x) = 1 > 0 \). We obtain then that \( \int_{r}^{x} (N - 2) \frac{dF(x)}{dx} \left( \begin{array}{c}
\frac{F(1)}{F(1)} \\
\frac{F(2)}{F(2)}
\end{array} \right)^{N-2} F(2)(x) F^x(u) d[-\varphi(x - u)] > 0 \). As above, we have also \( \frac{d\varphi^V(x)}{dx} \left( \begin{array}{c}
\frac{F(1)}{F(1)} \\
\frac{F(2)}{F(2)}
\end{array} \right)^{N-2} F(2)(x) F^x(u) \left( \begin{array}{c}
\frac{F(1)}{F(1)} \\
\frac{F(2)}{F(2)}
\end{array} \right)^{N-2} F(2)(x) F^x(u) \) \( \leq 1 \) and finally \( \psi^V(x) > 0 \).

In the same way as we have shown in the proof of Proposition 3.7 that the bid function \( \beta^V_R \) is regular and efficient, we conclude that the bid function \( \beta^V_R \) is regular and efficient. Similarly, we have furthermore \( S^A_+ = \emptyset \) so that \( U(v, \cdot) \) is continuous on \( [r, \infty) \) and \( (r, b) \subseteq S^{NA} \).

We now have to check that \( \beta^V_R \) is an equilibrium. Below we will use the facts that: 1/ if \( V \) is NDARA, then \( V(c) = E[V(Y)] \) implies \( V(c) \geq E[V(Y)] \) for all stochastic variable \( Y \) (see McAfee and Vincent [32]); 2/ (28) guarantees that the first-order condition (27a) holds for any \( v \in T_r \). Let us now show that \( \beta^V_R(v) \in S^{\max}_U(v) \) for any \( v \in T \). For \( v \in U\left((r, r)\right) \), it is straightforward as with risk-neutral buyers. Consider now \( v \in T \setminus U\left((r, r)\right) \).

For a given \( b \in (r, b) \) and \( x \in [r, \bar{v}] \), we define \( \zeta \) by

\[
\begin{align*}
\psi^V(x - \zeta(x, b)) &= \int_{0}^{\bar{v}} \varphi(x - \zeta(x, b)) dH_2(u|b) \tag{59}
\end{align*}
\]
From the first-order condition (27a), we have ζ(β_{R1}^V)^{-1}(b, b) = b. Furthermore, the differentiation of (59) w.r.t. x yields
\[
\frac{\partial \zeta(x, b)}{\partial x} = 1 - \frac{\int_0^x \mathcal{V}'([x - u] \lor 0)dH_2(u|b)}{\mathcal{V}(x - \zeta(x, b))} \geq 1 - \frac{\int_0^\tau \mathcal{V}'([x - u] \lor 0)dH_2(u|b)}{\mathcal{V}(x - \zeta(x, b))} \geq 0
\]
where the last inequality comes from the fact 1/ above and the definition of ζ in (59).

Take b ∈ (r, \bar{b}) and let \( v'_1 = (\psi^\mathcal{V})^{-1}(b) \in [r, \bar{v}] \). From (26a) and since \( H_1(u|b) = 0 \) if \( u < v'_1 \), we have
\[
\frac{\partial U(v, b)}{\partial b} = \begin{cases} \text{resp.} \geq 0 & \text{if } b \geq \beta_{R1}^V(v) \text{ [resp. } b \leq \beta_{R1}^V(v)] \text{ or equivalently } v'_1 \geq v_1 \text{ [resp. } v'_1 \leq v_1], \text{ since } \psi^\mathcal{V} \text{ is strictly increasing.} \\
\end{cases}
\]
Since \( \zeta(\cdot, b) \) is increasing, we have also \( \mathcal{V}(v_1 - b) = \mathcal{V}(v_1 - \zeta(v'_1, b)) \leq [\text{resp.} \geq \text{if } b \geq \beta_{R1}^V(v) \text{ [resp. } b \leq \beta_{R1}^V(v)]\text{ from (60)}. \]

Since we have \( U(v, r) \geq U(v, -1) \) and \( U(v, b) = U(v, \bar{b}) \) for any \( b \geq \bar{b} \) (as with risk-neutral buyers), we obtain finally that \( \beta_{R1}^V(v) \in S_{U}^{\text{max}}(v) \) for any \( v \in T_r \) and then on the whole that \( \beta_{R1}^V \) is an equilibrium.

We conclude after noting that any efficient bid function is based solely on first valuations when \( x^* = \bar{v} \) so that the first-order condition (27a) leads finally to the expression (28) as the unique equilibrium candidate.

V Proof of Proposition 5.1

The dynamic game is one where actions are perfectly observable. From the one-shot deviation principle (see Fudenberg and Tirole [15]), it is thus sufficient to check that each type has no profitable deviation at each phase \( k = 0, \ldots, N - 2 \). As argued in Section 5, for phase \( N - 2 \), this comes from our analysis of second-price auctions and more precisely from Remark 3.6. Consider now phase \( k \leq N - 3 \). If \( b_k \geq \bar{v} \), then it is straightforward that exiting immediately is a dominant strategy. Consider now that \( b_k < \bar{v} \). We first note that all bids above \( b_k \lor v_1 \) raise the same expected revenue: the point is that this deviation has an impact on the bidding history only if the given buyer has the lowest valuation among the remaining active \( N - k \) bidders. However, in those events, the (deviant) bidder will not receive any unit in the continuation game (he will drop immediately if another bidder quit
the auction before him) such that his payoff is fixed to zero. We are thus left with the one-shot deviations \( b \) with \( b < b_k \lor v_1 \) or equivalently \( b \in [b_k, v_1) \). Let us show that such a deviation can not be strictly profitable for \( k \leq N - 3 \).

Among the given buyer’s opponents, let bidder A [resp. B] be the one with the highest [resp. second-highest] first valuation and denote \( v^A \) [resp. \( v^B \)] his type. Let \( b^{ex} \) denote the realization of the bid of the extra bidder. Let us compare the expected payoff from the (one-shot) deviation \( b \) with the expected payoff from the alternative following deviation: bid according to the strategy \( \beta \) up to phase \( N - 3 \), while in phase \( N - 2 \) exit immediately if \( b_{k+1} \in (b, v_1] \) and bid according to the strategy \( \beta \) otherwise. Let us call this strategy \( \beta^*(v) \).

We first consider the events where bidder B wins the first unit under deviation \( b \)’s scenario. If \( v_1 \leq v^A_1 \) then his payoff is null with the deviation \( b \) (so that he could only have been weakly better off with the strategy \( \beta^*(v) \) which guarantees a positive payoff since it does not involve any bid above his first valuation at any phase). If \( v_1 > v^A_1 \), then his payoff is equal to \( (v_1 - b^{ex} \lor v^A_1) \lor 0 \). On the contrary, with the strategy \( \beta^*(v) \), then bidder A would have won the first unit with probability one and the given buyer’s expected payoff would have been at least \( (v_1 - b^{ex} \lor v^A_1 \lor v^B_1) \lor 0 \) which is greater than \( (v_1 - b^{ex} \lor v^A_1) \lor 0 \).

On the whole, we have shown that the given buyer would have been weakly better off by using the strategy \( \beta^*(v) \) than the deviation \( b \) in the events where bidder B wins the first unit at phase \( N - 2 \) under deviation \( b \)’s scenario.

We consider then the events where the given buyer wins at phase \( N - 2 \) under deviation \( b \)’s scenario, which implies that \( b_{k+1} \leq b \). In such events, the given buyer would have also won at phase \( N - 2 \) under deviation \( \beta^*(v) \)’s scenario since both strategies coincide on such bidding histories.

We are left finally with the events where bidder A wins at phase \( N - 2 \) under deviation \( b \)’s scenario. If \( v_1 \leq v^B_1 \) then the given buyer’s payoff is null with the deviation \( b \) (so that he could only have been weakly better off with the strategy \( \beta^*(v) \)). Consider now the events where \( v_1 > v^B_1 \). Under strategy \( \beta^*(v) \)’s scenario, bidder A would have also won the first unit with probability one (the deviation \( \beta^*(v) \) has an impact on the bidding history compared to the deviation \( b \) only when \( b_{k+1} \in (b, v_1] \), i.e. in the cases where the given buyer will precisely let bidder A win with probability one at the phase \( N - 2 \)). Both strategies yield thus the same expected payoff in those events.

On the whole deviation \( \beta^*(v) \) (weakly) outperforms deviation \( b \). We conclude after noting that strategy \( \beta^*(v) \) is (weakly) outperformed by the strategy \( \beta \) since the only difference

---

\(^{48}\)Note that we use implicitly in our argument that there is a null probability that two opponent bidders (playing the strategy \( \beta \)) exit at the same price for phases \( k = 0, \ldots, N - 3 \) and a null probability that the other bidder exit immediately at phase \( N - 2 \).
is from phase $N - 2$ where strategy $\beta$ is optimal as argued above from Remark 3.6.