On minimal ascending auctions with payment discounts*

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Abstract

The literature on ascending combinatorial auctions yields conflicting insights regarding the possibility to implement the Vickrey payoffs for general valuations. We introduce the class of minimal ascending auctions, a class which allows one to disconnect the final payments from the final bids but which prohibits the raising of the price vector of a provisionally winning bidder. We first establish the impossibility of implementing the Vickrey payoffs for general valuations. Second, we propose a minimal ascending auction that yields a bidder-optimal competitive equilibrium thanks to payment discounts.

Keywords: ascending auctions, combinatorial auctions, bidder-optimal competitive equilibrium, Core-selecting auctions, Vickrey payoffs

JEL classification: C70, D44

1 Introduction

Ascending auctions may be preferred to their sealed-bid counterparts due to various reasons: bidders’ reluctance to reveal their private preferences (Rothkopf et al. [22]), the possibility of acquiring information in the course of the auction (Compte and Jehiel [4]), interdependent valuations (Krishna [12]) or bidding complexity (Porter et al. [21]) such that dynamic formats allow bidders to learn the relevant packages they should bid on.¹ Developing ascending counterparts of sealed-bid formats is thus of prime interest in the multi-object auction literature.

For general valuations, de Vries et al. [7] (henceforth dVS&V) establish the impossibility of implementing the Vickrey payoffs under a general class of ascending auctions allowing both non-linear and personalized prices. On the contrary, under a seemingly mild extension of dVS&V’s formalization, Mishra and Parkes [17] (henceforth M&P) propose a class of ascending auctions which achieve the Vickrey payoffs for general valuations. M&P’s solution

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¹See the working paper version [15] for additional details and references on those motivations.
relies on a new ingredient, payment discounts: at the very last stage, the final payments of the bidders do not coincide with their final bids but are determined from the final price vector and the corresponding demand sets (or bids). Per se, payment discounts and more generally the possibility of disconnecting bidders’ final payments from their final bids do not break the desirable features of ascending formats that were briefly brought up in the first paragraph. Nevertheless, M&P’s solution relies on the unappealing feature that the auctioneer may still raise the price vector after a competitive equilibrium has been reached. At this stage where the efficient assignment is thus known, the final payoff of a given bidder in an auction implementing the Vickrey payoffs will not depend anymore on his own further reports but only on the further reports of his opponents. Consequently, once the price vector has reached a competitive equilibrium, bidders are then indifferent between all possible reports in M&P’s Vickrey auction.\(^2\) More generally, the price vector of a provisionally winning bidder may raise in M&P’s solution. The definitions of ascending auctions proposed by dVS&V and Gul and Stacchetti [10] to formalize their impossibility results do not exclude this feature. However, it never occurs in the formats they propose for their possibility results.\(^3\) This invites us to reconsider what should be taken as an ascending auction by adding a *minimality* property requiring that only losing bidders according to the current set of bids can face a price increase, while allowing for payment discounts.

This note is organized as follows. Section 2 introduces the model and the notation. Section 3 is devoted to our impossibility result: any minimal ascending auction fails to implement the Vickrey payoffs for general valuations. Nevertheless, payment discounts appear to be fruitful insofar as they allow one to implement a bidder-optimal competitive equilibrium as established in Section 4 where a new class of ascending combinatorial auctions is proposed. Furthermore we show how our payment discount rule relates to the one proposed by M&P. Section 5 briefly discusses the theoretical status of this new ascending format. All proofs are relegated to Appendices A-D.

\(^2\)This unappealing feature has been mentioned by M&P and is actually a corollary of the foundation of the Vickrey payoff for a given bidder that depends solely on the externality he imposes on his opponents which does not depend anymore on his own preferences once the final efficient assignment has been found. A similar unappealing feature occurs in Mishra and Parkes’ [18] descending Vickrey auctions once a “competitive equilibrium of the main economy” has been found and also in Ausubel’s [1] dynamic auction with multiple parallel price paths, e.g. once the Vickrey payoff of some bidders has been computed. In the same vein, Jehiel and Moldovanu [11] point a similar unappealing indifference in Mezzetti’s [16] efficient mechanism with interdependent valuations.

\(^3\)Gul and Stacchetti [10] establish the impossibility of implementing the Vickrey payoffs through ascending auctions with linear and anonymous prices, even if valuations are satisfying the gross substitutes condition. On the other hand, they propose an ascending auction that yields bidders’ most preferred Walrasian equilibrium, generalizing thus Demange et al. [8] from the unit-demand framework to general preferences satisfying the gross substitutes condition.
2 The package model

There is a finite set of bidders $\mathcal{N} = \{1, \ldots, N\}$, a single seller indexed as agent 0 and a finite set of indivisible goods $\mathcal{G}$. The set of feasible assignments of the goods is denoted as follows: $\mathbf{A} := \{A \in (2^\mathcal{G})^{N+1} : i \neq j \Rightarrow A_i \cap A_j = \emptyset \text{ and } \bigcup_{i=0}^{N} A_i = \mathcal{G}\}$. Each bidder $i \in \mathcal{N}$ has a non-negative integer valuation for each set of goods $H \subseteq \mathcal{G}$, denoted by $v_{i,H} \in \mathbb{N}$, and with $v_{i,\emptyset} = 0$. The seller values the goods at zero. We assume free disposal, i.e. $H \subseteq H'$ implies $v_{i,H} \leq v_{i,H'}$. Preferences are quasi-linear: a bidder $i$ who consumes $H \subseteq \mathcal{G}$ and makes a payment of $p_{i,H} \in \mathbb{R}_+$, which denotes the price of bundle $H$ for bidder $i$, receives a net payoff of $v_{i,H} - p_{i,H}$. For a given assignment $A$, the total welfare equals then $\sum_{i \in \mathcal{N}} v_{i,A_i}$. At (personalized) price vector $p \in \mathbb{R}_+^{2^{\mathcal{G}} \times \mathcal{N}}$, the demand set (or bids) of bidder $i$, the supply set of the seller w.r.t. prices and the supply set of the seller w.r.t. bids are defined respectively as follows:

$$D_i(p; v) := \arg\max_{H \subseteq \mathcal{G}} (v_{i,H} - p_{i,H}); \quad L(p) := \arg\max_{A \in \mathbf{A}} \sum_{i \in \mathcal{N}} p_{i,A_i}; \quad \text{and } L^*(p) := \arg\max_{A \in \mathbf{A} \mid A \in \{0, D_i(p; v)\}} \sum_{i \in \mathcal{N}} p_{i,A_i}.$$ 

At a given price vector $p \in \mathbb{R}_+^{2^{\mathcal{G}} \times \mathcal{N}}$, we say that bidder $i$ is a winning bidder [resp. a losing bidder] if $A \in L^*(p) \Rightarrow A_i \in D_i(p; v)$ [resp. $\exists A \in L^*(p)$ such that $A_i \notin D_i(p; v)$].

**Definition 1** Price vector $p \in \mathbb{R}_+^{2^{\mathcal{G}} \times \mathcal{N}}$ and assignment $A \in \mathbf{A}$ form a competitive equilibrium (CE) [quasi-competitive equilibrium (quasi-CE)] if $A_i \in D_i(p; v)$ for every bidder $i \in \mathcal{N}$ and $A \in L(p) \mid [A \in L^*(p)]$. Price $p$ is then called a CE [quasi-CE] price vector.

For a given price vector $p \in \mathbb{R}_+^{2^{\mathcal{G}} \times \mathcal{N}}$, let $\gamma^*(p) \in \mathbb{R}^{N+1}$ denote the corresponding payoff vector such that $\gamma^*_0(p) = \max_{A \in \mathbf{A}} \sum_{i \in \mathcal{N}} p_{i,A_i}$ and $\gamma^*_i(p) = \max_{H \subseteq \mathcal{G}} (v_{i,H} - p_{i,H})$ for $i \in \mathcal{N}$.

**Definition 2** A price vector $p$ is semi-truthful if for each $i \in \mathcal{N}$ there exists $\gamma_i \in [0, v_{i,G}]$ such that $p_{i,H} = \max \{v_{i,H} - \gamma_i, 0\}$ for any $H \subseteq \mathcal{G}$. In such a case, we have $\gamma_i = \gamma_i^*(p)$.

If $p$ is a CE price vector then $\gamma^*(p)$ is called a CE payoff. In the following, let $CEP(\mathcal{N}, v)$ denote the set of CE payoffs.

**Definition 3** The set of bidder-optimal CE payoffs, denoted by $BOCE(\mathcal{N}, v)$, is the set containing the elements $\gamma \in CEP(\mathcal{N}, v)$ such that there exists no CE payoff $\gamma' \in CEP(\mathcal{N}, v)$ with $\gamma'_i \geq \gamma_i$ for all $i \in \mathcal{N}$ and such that at least one inequality is strict.

Let $w$ denote the characteristic function associated to this assignment problem: it is defined by $w(S) := \max_{A \in \mathbf{A}} \sum_{i \in S} v_{i,A_i}$ for any $S \subseteq \mathcal{N}$.

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4We say that a price vector $p$ is linear if $p_{i,H} + p_{i,H'} = p_{i,H \cup H'}$ for any $i \in \mathcal{N}$, $H, H' \subseteq \mathcal{G}$ with $H \cap H' = \emptyset$. 

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Definition 4 We say that bidders are submodular (BAS) if the characteristic function \( w \) is submodular, i.e., if \( w(M \cup \{i\}) - w(M) \geq w(M' \cup \{i\}) - w(M') \) for all \( M \subseteq M' \subseteq \mathcal{N} \) and all \( i \in \mathcal{N} \), and that bidders are substitutes (weak BAS) if \( w(\mathcal{N}) - w(\mathcal{N} \setminus M) \geq \sum_{i \in M} [w(\mathcal{N}) - w(\mathcal{N} \setminus \{i\})] \) for all \( M \subseteq \mathcal{N} \).

The BAS condition implies the weak BAS condition as reflected by our terminology. Those conditions are key sufficient conditions in the literature in order to derive possibility results.

A key payoff vector is the Vickrey payoff vector, denoted by \( \gamma_V \in \mathbb{R}^{\mathcal{N}+1} \), which is defined by \( \gamma_V^i := w(\mathcal{N}) - w(\mathcal{N} \setminus \{i\}) \) for any \( i \in \mathcal{N} \) and \( \gamma_V^0 := w(\mathcal{N}) - \sum_{i \in \mathcal{N}} \gamma_V^i \). Finally we define the set of Core payoffs, denoted by \( \text{Core}(\mathcal{N}, v) \), related to the characteristic function \( w \):

\[
\text{Core}(\mathcal{N}, v) := \left\{ \gamma \in \mathbb{R}^{\mathcal{N}+1}_+ \mid (a) \sum_{i \in \mathcal{N} \cup \{0\}} \gamma_i \leq w(\mathcal{N}); \quad (b) \quad (\forall S \subseteq \mathcal{N}) \quad w(S) \leq \sum_{i \in S \cup \{0\}} \gamma_i \right\}.
\]

(a) is the feasibility condition whereas the inequalities (b) mean that the payoffs are not blocked by any coalition with the seller and the subset of bidders \( S \).

Proposition 2.1 (Bikhchandani and Ostroy [3])

- \( \text{Core}(\mathcal{N}, v) = \text{CEP}(\mathcal{N}, v) \neq \emptyset \)
- The weak BAS condition is equivalent to the Vickrey payoff vector being a CE payoff vector, which is also equivalent to the set of bidder-optimal CE payoffs being a singleton. In such a case they coincide: \( \text{BOCE}(\mathcal{N}, v) = \{\gamma_V\} \).

For any \( \gamma \in \mathbb{R}^{\mathcal{N}+1}_+ \), we define the price vector \( \mathcal{P}(\gamma) \in \mathbb{R}_{+}^{2^G \times \mathcal{N}} \) in the following way: \( [\mathcal{P}(\gamma)]_{i,H} := \max \{v_{i,H} - \gamma_i, 0\} \) for any \( i \in \mathcal{N} \) and \( H \subseteq G \). In order to prove the inclusion \( \text{Core}(\mathcal{N}, v) \subseteq \text{CEP}(\mathcal{N}, v) \), Bikhchandani and Ostroy [3] have shown that any point \( \gamma \) in the Core can be priced by \( \mathcal{P}(\gamma) \), which is thus a semi-truthful CE price vector.

3 Ascending Vickrey auctions

Similarly to dVS&V, we define a class of ascending auctions by generalizing Gul and Stacchetti’s [10] definition to allow for non-linear and personalized prices. Contrary to dVS&V, we do not require that finals payments should coincide with final bids. In particular, our definition allows for payment discounts as in M&P. However, we strengthen dVS&V’s definition by requiring that only losing bidders according to the current set of bids can face a price increase.

\footnote{For \( M = \{l_1, \ldots, l_M\} \), the summation of the inequalities \( w(\mathcal{N}) - w(\mathcal{N} \setminus \{l_k\}) \leq w(\mathcal{N} \setminus \cup_{i=1}^{k-1} \{l_i\}) - w(\mathcal{N} \setminus \cup_{i=1}^{k} \{l_i\}) \) (for \( k = 1, \ldots, M \)) yields \( \sum_{i \in M} [w(\mathcal{N}) - w(\mathcal{N} \setminus \{i\})] \leq w(\mathcal{N}) - w(\mathcal{N} \setminus M). \)}
A price path is a right-continuous function \( P : [0, 1] \to \mathbb{R}_+^G \times N \) such that,

1. \( P_{i,0}(t) = 0 \),
2. \( H \subseteq H' \) implies \( P_{i,H}(t) \leq P_{i,H'}(t) \). For each bundle \( H \subseteq G \), interpret \( P_{i,H}(t) \) as the price seen by bidder \( i \) for bundle \( H \) at “time” \( t \). A price path is ascending if for any \( i \in N \) and \( H \subseteq G \), the function \( P_{i,H}(t) \) is nondecreasing in \( t \). Let \( \mathcal{P} \) denote the set of all ascending price paths.

### Definition 6

A minimal ascending auction is a mapping \((\pi, \xi, \omega)\) which assigns to each valuation profile \( v \in \mathbb{R}_+^G \times N \) an ascending price path \( \pi(v) \in \mathcal{P} \), a final vector of payments \( \xi(v) \in \mathbb{R}_+^N \) and a final assignment \( \omega(v) \in A \) such that:

1. for all valuation profiles \( v, v' \in \mathbb{R}_+^G \times N \) and any \( \hat{t} \in [0, 1] \), if \( D_i(\pi(v))(t); v) = D_i(\pi(v'))(t); v' \) for any \( t \in [0, \hat{t}] \) and \( i \in N \), then \( \pi(v)(t) = \pi(v')(t) \) for any \( t \in [0, \hat{t}] \);
2. for all valuation profiles \( v, v' \in \mathbb{R}_+^G \times N \), if \( D_i(\pi(v))(t); v) = D_i(\pi(v'))(t); v' \) for any \( t \in [0, 1] \) and \( i \in N \), then \( \xi(v) = \xi(v') \) and \( \omega(v) = \omega(v') \);
3. [Minimality property] if \( \pi(v)(t')_{i,H} \geq \pi(v)(t)_{i,H} \) for some \( H \subseteq G \), \( i \in N \) and \( t' > t \), then there exists \( \hat{t} \in [t, t'] \) and \( A \in L^*(\pi(v)(\hat{t})) \) such that \( A_{i} = \emptyset \neq D_i(\pi(v)(\hat{t}); v) \).

Condition (i) formalizes that price adjustments are responsive only to the demand sets reported by the bidders along the price path. Up to Section 5, we assume implicitly that bidders are reporting their demand sets truthfully. The unique restriction imposed on the final allocation by condition (ii) is that it depends solely on the information revealed along the price path. In particular, final payments may not correspond to final bids, i.e. we may have \( \xi(v) \neq \pi(v)(1)_{i,\omega(v)} \). However, for our positive results in next section and as in M&P, the final allocation will be determined solely by the information contained at the moment where the auction stops: the functions \( \xi \) and \( \omega \) will depend on \( v \) solely through the final vector of prices \( \pi(v)(1) \) and the corresponding demand sets \( D_i(\pi(v)(1); v) \), \( i \in N \). Without condition (iii), Definition 6 would allow one to implement any outcome, e.g. the Vickrey payoffs, in a straightforward way: the idea is to recover fully bidders’ preferences by raising the prices up to their valuations in a first stage (which is feasible with personalized non-linear prices) and then to pick the desired outcome in a second stage. However, such a solution would break the analogy with the English auction and more generally stands in conflict with what we have in mind under the terminology “ascending auction”. In particular, according to Rothkopf et al.’s [22] perspective of bidders’ reluctance to reveal their true preferences, such a solution is obviously not satisfactory. dVS&V’s restriction that final payments should correspond to final bids has no clear status according to the aforementioned privacy preserving perspective. This later perspective suggests, on the contrary, to strengthen the linkage between demand revelation and price adjustments. This is precisely what the minimality property does by imposing that only the price vector of
losing bidders can be pushed up. It means in particular that a given bidder’s price vector is definitely frozen once the empty set belongs to his demand set. To the best of our knowledge, the only so-called “ascending auctions” in the literature that fail the minimality property are some of the auctions proposed by M&P.

**Proposition 3.1** There is no minimal ascending auction that yields the Vickrey payoffs for general valuations.

We emphasize the strength of our impossibility result since almost all auctions that appeared in the literature under the terminology ‘ascending auction’ satisfy our definition. Apart from M&P, we are aware of only two exceptions: Ausubel’s [1] dynamic auction for heterogenous goods with multiple parallel price paths in order to implement Vickrey payoffs under private values and substitutable preferences while maintaining linear prices and Perry and Reny’s [20] dynamic auction for homogenous goods under interdependent valuations in order to implement an efficient assignment.\(^6\)

The proof relies on a class of preferences with two goods and three bidders and which is commonly studied in the literature (see, e.g., Krishna and Rosenthal [13]): there is one ‘global’ bidder who values solely the bundle and two ‘local’ bidders who are interested in only one object. The impossibility result does not hold if the Vickrey payoff vector is guaranteed to be a CE payoff vector such that Corollary 4.4 would apply. Proposition 3.1 can also be interpreted as saying that the computation of the Vickrey payoffs with an ascending auction would require the violation of the minimality property. We obtain thus that the solutions proposed by M&P to implement the Vickrey payoffs for general valuations have to fail the minimality property for some realization of the valuations. This failure is illustrated in the example used throughout next section.

### 4 A class of ascending bidder-optimal CE selecting auctions

Our construction is a variation on a well-known building block, labeled below as QCE-invariant ascending auctions. It works as follows: at each stage, bidders are asked to report their demand sets for the current price vector; the seller chooses an assignment that maximizes her (provisional) revenue according to those demand sets; then for bidders that do not obtain a bundle in their demand set the corresponding prices are increased by

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\(^6\)In the same way as payment discounts per se do not stand in conflict with what should be viewed as an ascending auction, we should not dispose a priori of the idea of multiple price paths. However, it may stand in conflict with what we have captured under the minimality property defined for ascending auctions with a single price path: in particular, we want to prevent us to ask a given bidder further information about his preferences once we have reached a point where we are able to compute his final assignment (see also footnote 2).
Definition 7 A QCE-invariant ascending auction is defined as follows:

(S0) The auction starts at round one, \( t := 1 \), with the zero price vector, \( p^1 := 0 \).

(S1) In round \( t \) of the auction, with price vector \( p^t \):

(S1.1) Collect the demand sets of the bidders \((D_i(p^t; v))_i \in \mathcal{N}\), select a temporary winning assignment \( A^t \in L^*(p^t) \), and let \( \mathcal{L}^t := \{i \in \mathcal{N}| A^t_i = \emptyset \neq D_i(p^t; v)\} \).

(S1.2) If \( \mathcal{L}^t = \emptyset \), then go to Step (S2) with \( T := t \).

(S1.3) Else, select a non-empty subset \( S^t \subseteq \mathcal{L}^t \). Let \( p^{t+1}_{i, H} := p^t_{i, H} + 1 \) for any \( i \in S^t \) and \( H \in D_i(p^t; v) \), and let \( p^{t+1}_{i, H} := p^t_{i, H} \) otherwise. Move to the next round, \( t := t + 1 \), and repeat from Step (S1).

(S2) The auction ends with the final assignment of the auction being \( A^T \) and the final payment for each bidder \( i \in \mathcal{N} \) being \( p^T_{i, A^T_i} \), where \( p^T \) is the final price vector.

The dynamic of the algorithm is illustrated in Table 1, which contains an example with three identical items and five bidders and which is used throughout this section to illustrate our definitions and our results.

The auctions proposed by Ausubel and Milgrom [2], Parkes [19] and dVS&V are QCE-invariant ascending auctions. QCE-invariant ascending auctions belong to the class of “uQCE-invariant(0) auction for the main economy” considered by M&P. However, the other inclusion fails: in our definition, the set of bidders that may face a price increase is \( \mathcal{L}^t \), which is a subset of the set of losing bidders,\(^7\) while M&P are allowing any price increase for a given bidder provided that the empty set is not in his demand set. This is an illustration of how the minimality property may fail in the auctions proposed by M&P. The property emphasized in the aforementioned papers is that QCE-invariant ascending auctions yield the Vickrey payoffs if the BAS condition holds. Our analysis relies crucially on another key property: those auctions terminate in a CE semi-truthful price vector for general valuations. Useful (rather well-known) properties are gathered in next lemma (see the working paper version [15] for the proof. Many elements already appeared in Ausubel and Milgrom [2]).

Lemma 4.1 A quasi-CE semi-truthful price vector is a CE. In a QCE-invariant ascending auction, the price \( p^t \) at each round \( t \) is semi-truthful. The algorithm ends in a finite number of rounds. As a corollary, \( p^T \) is a CE semi-truthful price vector.

\(^7\)If \( L^*(p^t) \) is a singleton, then \( \mathcal{L}^t \) coincides with the set of losing bidders.
The seller's provisional revenue is approached from above insofar as bidders' provisional payoffs are increasing in $t$.

Although QCE-invariant ascending auctions stop once a CE has been reached and that the set of CE payoffs is approached from above insofar as bidders' provisional payoffs $\gamma^s_i(p^t)$ are decreasing over time (since $p^t$ is increasing in $t$), the final payoff vector may not be a bidder-optimal CE payoff vector in those auctions, as it is illustrated in Table 1. The rest of this section consists mainly in modifying slightly the class of QCE-invariant ascending auctions in order to implement a payoff vector in BOCE($N, v$) for general valuations.

In order to build some discount rules that will come on the top of QCE-invariant ascending auctions, we first introduce some additional notation and definitions. For any vector $e \in \mathbb{R}^N_+$ and $p \in \mathbb{R}^{2d \times N}_+$, let $\beta(e; p) = p'$ denote the price vector such that $p'_{i,H} = $...
max\{p_i,H - e_i, 0\} for any \(i \in \mathcal{N}\) and \(H \subseteq \mathcal{G}\).

**Definition 8** A vector \(e \in \mathbb{R}_+^\mathcal{N}\) is called an admissible discount with respect to a quasi-CE price vector \(p \in \mathbb{R}_+^{\mathcal{G} \times \mathcal{N}}\) if there exists \(A^* \in \mathcal{A}\) such that \(e_i \leq p_iA^*_i\) for any \(i \in \mathcal{N}\), \((p, A^*)\) forms a quasi-CE and

\[
A^* \in \operatorname{Arg}\max_{A \in \mathcal{A} | \forall i \in \mathcal{N}} \left[ \sum_{i \in \mathcal{N}} [\beta(e;p)]_{i,A_i} \right].
\]

(1)

Let \(\mathcal{E}(p)\) denote the set of admissible discounts w.r.t. \(p\) and \(\Gamma(p)\) the corresponding set of payoff vectors, i.e. \(\Gamma(p) := \{\gamma \in \mathbb{R}_+^{\mathcal{N}+1} | \exists e \in \mathcal{E}(p) \text{ such that } \gamma = \gamma^*(\beta(e;p))\}\).

The key property of admissible discounts is that they “preserve quasi-CE” as formalized below. In the perspective of QCE-invariant ascending auctions where the algorithm stops once a quasi-CE price vector \(p\) has been reached, it means that if the price vector dynamic had been different so that it would have reached the price \(\beta(e;p)\) where \(e \in \mathcal{E}(p)\), then the auction would have stopped at \(\beta(e;p)\).

**Proposition 4.1** If \((p, A)\) is a quasi-CE and \(e \in \mathcal{E}(p)\), then \((\beta(e;p), A)\) is a quasi-CE. Furthermore, \(\gamma^*_i(\beta(e;p)) = \gamma^*_i(p) + e_i\) for any \(i \in \mathcal{N}\).

After noting that if \(p\) is a semi-truthful price vector, then \(\beta(e;p)\) is also a semi-truthful price vector,\(^9\) we conclude from Proposition 4.1 that applying an admissible discount from a quasi-CE semi-truthful price vector \(p\) yields a CE payoff vector that yields larger payoffs for the bidders compared to the payoff vector \(\gamma^*(p)\).

**Definition 9** A maximal discount rule \(\delta_{\text{max}}(.)\) is a function that assigns to any quasi-CE price vector \(p\) a price vector \(\delta_{\text{max}}(p)\) such that: 1) there exists \(e^* \in \mathcal{E}(p)\) such that \(\delta_{\text{max}}(p) = \beta(e^*;p)\), 2) there is no \(e \in \mathcal{E}(p)\) such that \(e_i \geq e^*_i\) for any \(i \in \mathcal{N}\) while \(e_i > e^*_i\) for some \(i\).

From Proposition 4.1, maximal discount rules are thus those that select (strict) bidder Pareto-optima in the set \(\Gamma(p)\), where \(p\) is the price vector from which the discount rule is applied. Next proposition states that such a rule implements actually a bidder-optimal CE payoff vector once \(p\) is semi-truthful.

**Proposition 4.2** For any maximal discount rule \(\delta_{\text{max}}\): if \(p\) is a quasi-CE semi-truthful price vector, then \(\gamma^*(\delta_{\text{max}}(p)) \in BOCE(\mathcal{N}, v)\).

\(^9\)We have \(\beta(e;p) = \mathcal{P}(\gamma^*(p)+\epsilon)\) since \(\max\{max\{v_i,H - \gamma^*_i(p),0\} - e_i,0\} = max\{v_i,H - (\gamma^*_i(p) + e_i),0\}\).
To prove Proposition 4.2, it is sufficient to show that bidder Pareto-optima in the set $\Gamma(p)$ are belonging to the set $BOCE(\mathcal{N}, v)$. This is done by establishing that $\Gamma(p)$ coincides with the set of all CE payoffs that are larger (according to bidders’ payoffs) than the payoff vector $\gamma^*(p)$, i.e., for any (quasi-)CE semi-truthful price vector $p$, we have $\Gamma(p) = [\gamma^*(p)]^{-1} \cap CEP(\mathcal{N}, v)$.\(^{10}\) If $\Gamma(p) \cap BOCE(\mathcal{N}, v)$ is a singleton, then there is a unique candidate for $\delta_{\max}(p)$. Otherwise, there are various candidates to be a solution. In Table 1, from the CE price vector reached at round 8 (where our QCE-invariant ascending auction stops), we have $\Gamma(p^8) \cap BOCE(\mathcal{N}, v) = \{(6, x, 5 - x, 0, 0, 0) | x \in [2, 3]\}$. We pick then the symmetric solution $(6, 2.5, 2.5, 0, 0, 0)$.

We now establish the links between maximal discount rules and the discount rule proposed by M&P. For any quasi-CE price vector $p$, let $e_i(p) := \max_{e \in E(p)} e_i$ for any $i \in \mathcal{N}$ and let $\delta_{MP}(p) := \beta(e(p); p)$. In a nutshell, $e_i(p)$ corresponds to the greatest payoff increase that bidder $i$ may expect in a maximal discount rule from the quasi-CE $p$. The related discount rule $\delta_{MP}(.)$ is called the MP discount rule and is shown to coincide with the one proposed by M&P. Our construction gives then a more interpretable definition of their discount rule as the largest discount for a bidder so that the price vector remains a quasi-CE.

**Proposition 4.3** Consider a quasi-CE semi-truthful price vector $p$, the MP discount rule $\delta_{MP}(p)$ corresponds to the one proposed by M&P. Furthermore, we have:

$$\gamma_i^*(\delta_{\max}(p)) \leq \gamma_i^*(\delta_{\text{MP}}(p)) \leq \gamma_i^V, \text{ for any } i \in \mathcal{N}. \quad (2)$$

We now extend the class of QCE-invariant auctions by adding a payment discount stage after the auction dynamic stops.

**Definition 10** A QCE-invariant ascending auction with a maximal discount rule [with the MP discount rule] $\delta$ is a QCE-invariant ascending auction with Step (S2) replaced by (S2(δ)): “The auction ends with the final assignment of the auction being $A^T$ and the final payment for each bidder $i \in \mathcal{N}$ being $[\delta(p^T)]_{i,A^T}$, where $p^T$ is the final price vector and $\delta$ is a maximal discount rule [the MP discount rule $\delta_{MP}$].”

The explicit idea of a payment discount stage has been first introduced by M&P. The ‘clinching rule’ in Ausubel [1] is implicitly a payment discount stage: it corresponds to a payment discount that does not depend solely on the final price vector and the final bids, but on the whole price path. On the contrary, we emphasize that since the set $E(p)$ relies

\(^{10}\)For any payoff vector $\gamma \in \mathbb{R}^{N+1}$, let $[\gamma]^{-1} := \{\gamma' \in \mathbb{R}^{N+1} : \gamma'_i \geq \gamma_i \text{ for } i = 1, \ldots, N\}$ denote the set of payoffs than are larger than $\gamma$ for all bidders.
on bidders’ valuations only through the demand sets \( (D_i(p,v))_{i \in \mathcal{N}} \) reported at price \( p \), then the discount rules defined above depend solely on the current bids at \( p \).\footnote{This property seems desirable according to Compte and Jehiel’s [4] perspective where bidders may refine their valuations in the course of the auction process. Otherwise, payments could depend on out-of-date information about bidders’ demand sets. Note also that if the final price vector \( p^T \) is not semi-truthful in our QCE-invariant ascending auctions (due, e.g., to a chock affecting bidders’ preferences in the course of the auction), then the price vector after our discount rules is nevertheless guaranteed to be a quasi-CE.}

From Lemma 4.1 and Propositions 4.1 and 4.2, we obtain that the use of a maximal discount rule after a QCE-invariant auction yields a bidder-optimal CE payoff vector, which coincides with the Vickrey payoffs under the weak BAS condition (Proposition 2.1).

**Corollary 4.4** Any QCE-invariant ascending auction with a maximal discount rule implements a bidder-optimal CE payoff vector for general valuations. Under the weak BAS condition, it yields the Vickrey payoffs.

Payment discounts are actually necessary to implement a bidder-optimal CE payoff vector: in [15], we have shown that any ascending auction according to dVS&V’s definition, i.e. a class where final payments have to match final prices and which includes in particular any QCE-invariant ascending auction, fails to implement a bidder-optimal CE payoff vector for general valuations. Furthermore, this impossibility result holds even if the weak BAS condition prevails.

From (2), we also obtain that QCE-invariant ascending auctions with the MP discount rule (a class which has been already proposed by M&P) yield a payoff vector which lies between the Vickrey payoffs and some bidder-optimal CE payoff. The inequalities in (2) may be strict as illustrated in Table 1.

### 4.1 Implementing specific bidder-optimal CE payoffs?

At this stage, we have tried solely to implement a bidder-optimal CE payoff and not a specific payoff vector in \( BOCE(\mathcal{N},v) \) according to some selection rule.\footnote{See Day and Cramton [5] and Erdil and Klemperer [9] for proposals on this topic. Those works propose sealed-bid auctions but do not care about their implementation with some ascending auctions.} It brings us to the problem of determining the entire set of bidder-optimal CE payoffs from a given CE price \( p \). M&P and Lahaie and Parkes [14] have shown that the notion of universal competitive equilibrium (UCE)\footnote{A price vector \( p \) is called a *universal* CE if \( p \) is a CE price vector of the main economy containing the full set of bidders \( \mathcal{N} \) but also the ‘marginal’ economies, i.e. those containing \( \mathcal{N} - 1 \) bidders.} is the equilibrium notion that is necessary and sufficient to compute the Vickrey payoffs. In this perspective, Proposition 3.1 can be interpreted as saying that finding a UCE with an ascending auction would require a violation of the minimality property. This is illustrated in Table 1 where the price vectors of bidder 1 and 2 are increasing after rounds 8 and 9 although they are winning bidders with respect to the...
main economy. Similarly, we could conjecture that the set $BOCE(\mathcal{N}, v)$ can be computed from a UCE semi-truthful price vector. This conjecture is not true for general valuations as shown by the following example.

**Example** Consider four bidders and three goods $a, b$ and $c$. Consider that bidder 1 (resp. 2 and 3) values $V \geq 4$ any bundle containing the good $a$ (resp. $b$ and $c$) and 0 any other bundle. Consider that bidder 4 values 8 the bundle $abc$ and 0 any other bundle. The efficient assignment is the one that gives the goods $a, b$ and $c$ respectively to bidders 1, 2 and 3. Consider then the semi-truthful price vector $p$ characterized by $p_{1,a} = p_{2,b} = p_{3,c} = 4$ and $p_{4,abc} = 8$. $\gamma^*(p) = (12, V - 4, V - 4, V - 4, 0)$. $p$ is always a UCE price vector. However, the set $BOCE(\mathcal{N}, v)$ depends strictly on $V$: e.g. the payoff vector $(8, V, V, V, 0) \in BOCE(\mathcal{N}, v)$ if and only if $V \geq 8$.

Since the knowledge of the set $BOCE(\mathcal{N}, v)$ implies the knowledge of the Vickrey payoffs,$^{14}$ then it means that we need to explore bidders’ preferences strictly more than with a UCE price vector. In other words, the computation of the entire set of bidder-optimal CE payoffs would require a greater violation of the minimality property than in M&P where the price path ends once a UCE has been reached.

## 5 Conclusion on incentives

If the weak BAS condition holds, then we obtain from Lemma 4.1 and Propositions 4.2 and 4.3 that QCE-invariant ascending auctions with the MP discount rule yield the Vickrey payoffs. At first glance and regarding strategy-proofness, those later formats seem equivalent to the bidder-optimal CE selecting auctions we propose. It is true that if bidders were constrained to use a single identifier, then bidding truthfully would be an equilibrium in both kinds of auctions and would yield the same outcome. However, this is no longer true regarding a larger strategy-proofness perspective where bidders are free to use any identifiers (or shills) as they want: our proposal is robust to shills while QCE-invariant ascending auctions with the MP discount rule are not. This can be illustrated by coming back to Table 1. Suppose that the true preferences are such that bidders 1 and 2 form actually a single bidder A who is valuing two items $V > 5$ and zero any smaller bundle, while bidders 3, 4 and 5 remain unchanged. The weak BAS condition then holds and the Vickrey payoff vector is $(6, V - 5, 0, 0, 0)$ while the efficient assignment consists in giving two items to bidder A and one item to either 3 or 4. From Table 1, we obtain that bidder A may strictly benefit from using two identifiers in a QCE-invariant ascending auctions

$^{14}$For a given bidder, the Vickrey payoff corresponds to the upper bound of his payoffs among the set $BOCE(\mathcal{N}, v)$ (Bikhchandani and Ostroy [3]).
with the MP discount rule. The general argument follows from Day and Milgrom [6] who show that efficient auctions that are robust to shills are those that end in the Core. The crucial point is that it should end in the Core not solely for the true preferences but for any possible set of reports as our proposal does. Under the perspective where bidders strategy space includes also the possibility of using shills, strategy-proofness would thus require to implement both a CE payoff and the Vickrey payoff. If the weak BAS condition fails, there is thus no ‘strategy-proof’ mechanism. In such a case, the theoretical status of our proposal is to minimize the incentives to deviate from reporting truthfully the demand sets while giving no incentives to use shills.

References


Appendix

A Proof of Proposition 3.1

Consider two heterogeneous goods \(a\) and \(b\). Let \(V_1\) denote the set of bidder 1’s valuations such that \(v_{1,a} = v_{1,b} = 0\) and \(v_{1,ab} = x_1 \in \mathbb{N}\). Let \(V_2\) [resp. \(V_3\)] denote the set of bidder 2’s [resp. bidder 3’s] valuations such that \(v_{2,ab} = v_{2,a} = x_2 \in \mathbb{N}\) and \(v_{2,b} = 0\) [resp. \(v_{3,ab} = v_{3,b} = x_3 \in \mathbb{N}\) and \(v_{3,a} = 0\)]. Bidders’ valuation profiles are thus reduced to the three integers \(x_1, x_2\) and \(x_3\). Suppose that there exists a minimal ascending auction \((\pi, \xi, \omega)\) yielding the Vickrey payoffs on the domain \(V := V_1 \times V_2 \times V_3\). To alleviate notation, we set \(P \equiv \pi(v)\). Let \(\bar{t}(v) := 0\) if there exists \(i\) such that \(\emptyset \in D_i(P(0); v)\) and \(\bar{t}(v) := \sup\{t \in [0,1]|\emptyset \notin D_i(P(t); v), \forall i\}\) otherwise. For any \(t < \bar{t}(v)\), we have: 1) \(ab \in D_1(P(t); v), a \in D_2(P(t); v)\) and \(b \in D_3(P(t); v)\) (since \(P_{i,H}(t)\) is monotonic in \(H\)), 2) the provisional revenue raised by an assignment in \(L^*(P(t))\) is the maximum of \(P_{1,ab}(t)\) (with the corresponding assignment being the one which gives both goods to bidder 1) and \(P_{2,a}(t) + P_{3,b}(t)\) (with the corresponding assignment being the one which gives item \(a\) to bidder 2 and item \(b\) to bidder 3). For any \(i \in \mathcal{N}\) and \(H \subseteq \mathcal{G}\), let \(\tilde{P}_{i,H}(0) := 0\) and \(\tilde{P}_{i,H}(t) := \sup\{P_{i,H}(t')|t' < t\}\) for \(t \in (0,1]\). The information about valuations that is contained in the demand sets before time \(t \leq \bar{t}(v)\) (i.e. \(\{D_i(P(t'); v)\}_{i=1,2,3, t' < t}\)) is that
$x_1 > P_{1,ab}(t'), x_2 > P_{2,a}(t')$ and $x_3 > P_{3,b}(t')$ for any $t' < t$. From (i) in Definition 6, any valuations’ profile satisfying those conditions leads to the same price path up to $t$.

Suppose that $P_{2,a}(t) - \tilde{P}_{2,a}(t) > 2$ for $t \leq \bar{t}(v)$ and $v \in V$, then there exists an integer $s$ such that $\tilde{P}_{2,a}(t) < s < s + 1 < P_{2,a}(t)$. Take $x_3^*$ an integer such that $x_3^* > \tilde{P}_{3,b}(t)$. Consider the class of valuations’ profile where $x_1 > \max \{\tilde{P}_{1,ab}(t), s + 1 + x_3^*\}$, $x_2 \in \{s, s + 1\}$ and $x_3 = x_3^*$, a class which is compatible with the information revealed up to $t$. The auction cannot implement the Vickrey payoffs for sure since bidder 1’s Vickrey payment equals $x_2 + x_3$ which depends on the realization of $x_2$ among $\{s, s + 1\}$ which cannot be learned later (since $P_{2,a}(t) > s + 1$). We have thus raised a contradiction and we obtain then that $P_{2,a}(t) - \tilde{P}_{2,a}(t) \leq 2$ for any $t \leq \bar{t}(v)$. By symmetry, we have also $P_{3,b}(t) - \tilde{P}_{3,b}(t) \leq 2$ for any $t \leq \bar{t}(v)$. It implies in particular that $P_{2,a}(0) + P_{3,b}(0) \leq 4$. Let us now show that

$$P_{2,a}(t) + P_{3,b}(t) \leq P_{1,ab}(t) + 4 \text{ for any } t \leq \bar{t}(v).$$

If $P_{2,a}(t) + P_{3,b}(t) \leq 4$, we are done. Suppose now that $P_{2,a}(t) + P_{3,b}(t) > 4$ and let $\hat{t} := \sup \{t' < t \mid P_{2,a}(t') + P_{3,b}(t') < P_{2,a}(t) + P_{3,b}(t)\}$ (which is well defined since $P_{2,a}(0) + P_{3,b}(0) \leq 4$). From the minimality property, we obtain then that $P_{1,ab}(\hat{t}) \geq \tilde{P}_{2,a}(\hat{t}) + \tilde{P}_{3,b}(\hat{t})$. Since $P_{1,ab}(t) \geq \tilde{P}_{1,ab}(\hat{t})$ and $P_{2,a}(\hat{t}) + P_{3,b}(\hat{t}) \geq P_{2,a}(\hat{t}) + P_{3,b}(\hat{t}) - 4 = P_{2,a}(t) + P_{3,b}(t) - 4$ (where the last equality results from $P$ being right-continuous and the definition of $\hat{t}$), we obtain finally that (3) holds.

Suppose that $P_{1,ab}(t) - \tilde{P}_{1,ab}(t) > 4$ for $t \leq \bar{t}(v)$ and $v \in V$, then there exists an integer $s$ such that $\tilde{P}_{1,ab}(t) < s < s + 3 < P_{1,ab}(t)$. Let $x_2^*$ (resp. $x_3^*$) denote the smallest integer strictly above $\tilde{P}_{2,a}(t)$ (resp. $\tilde{P}_{3,b}(t)$). We have then $x_2^* + x_3^* \leq \tilde{P}_{2,a}(t) + \tilde{P}_{3,b}(t) + 2 \leq \tilde{P}_{1,ab}(t) + 6 < s + 6$ (where the second inequality comes from (3)). We have then either $x_2^* < s + 3$ or $x_3^* < s + 3$. Consider the case $x_2^* < s + 3$ (by symmetry, the case $x_3^* < s + 3$ can be treated similarly). Consider the class of valuations’ profile where $x_1 \in \{s + 2, s + 3\}$, $x_2 = x_2^*$ and $x_3 > \max \{\tilde{P}_{3,b}(t), s + 3 - x_2^*\}$, a class which is compatible with the information revealed up to $t$. The auction cannot implement the Vickrey payoffs for sure since bidder 3’s Vickrey payment equals $x_1 - x_2^*(\geq 0)$ which depends critically on the realization of $x_1$ among $\{s + 2, s + 3\}$ which cannot be learned later (since $P_{1,a}(t) > s + 3$). We have thus raised a contradiction and we obtain then that $P_{1,ab}(t) - \tilde{P}_{1,ab}(t) \leq 4$ for any $t \leq \bar{t}(v)$. In the same way as we have established (3), we obtain then

$$P_{1,ab}(t) - 4 \leq P_{2,a}(t) + P_{3,b}(t) \text{ for any } t \leq \bar{t}(v).$$

Since the functions $P_{i,H}(\cdot)$ are right-continuous, then we have $0 \in D_i(P(\bar{t}(v)); v)$ for at least one bidder, say $i^*$. Consider the price path for the subset of $V$ where $x_1 = 100$ and $x_2, x_3 \in \{97, 98, 99, 100\}$. Consider first the case $i^* = 1$. We have then $\tilde{P}_{1,ab}(\bar{t}(v)) \leq 100 \leq
where the second inequality comes from (1) and since \( A \in \gamma \). We have thus that either \( P_{2,a}(\tilde{t}(v)) < 54 \) and \( P_{3,b}(\tilde{t}(v)) > 42 \), or \( P_{3,b}(\tilde{t}(v)) < 54 \) and \( P_{2,a}(\tilde{t}(v)) > 42 \). By symmetry we can consider w.l.o.g. the first case. Then there is no way to learn precisely bidder 2’s valuation in the set \{97, 98, 99, 100\} since after \( P_{2,a}(\tilde{t}) \geq 97 \) for a given \( \tilde{t} > \tilde{t}(v) \), then any further price increase would be frozen due to the minimality property \( (P_{2,a}(\tilde{t}) + P_{3,b}(\tilde{t}) > 139 > P_{1,ab}(\tilde{t})) \). By setting only one price above 97, it is impossible to distinguish between four valuations.\(^{15}\) On the whole it is impossible to compute the Vickrey payment of bidder 3 which equals \( x_1 - x_2 \) and which thus depends critically on those four possible valuations. Consider then the case \( i^* = 2 \) (by symmetry the case \( i^* = 3 \) can be treated similarly). We have then \( P_{2,a}(\tilde{t}(v)) \geq 97 \) and \( P_{1,ab}(\tilde{t}(v)) < 104 \). From (3), we obtain that \( P_{3,b}(\tilde{t}(v)) < 11 \). As above, there is then no way to learn precisely bidder 3’s valuation in the set \{97, 98, 99, 100\} and thus to compute the Vickrey payoff of bidder 2. On the whole, we have raised a contradiction and we have thus shown that there is no minimal ascending auction that yields the Vickrey payoffs on the domain \( \mathbf{V} \).

B Proof of Proposition 4.1

Consider \( e \in \mathcal{E}(p) \). So there exists \( \mathcal{A}^* \in \mathcal{A} \) such that \( e_i \leq p_{i,A^*_i} \) for any \( i \in \mathcal{N} \), \((p, \mathcal{A}^*) \) forms a quasi-CE and (1) holds. In a first step, we show that \((\beta(e; p), \mathcal{A}^*) \) forms a quasi-CE. \( \mathcal{A}^* \in D_i(p; v) \) implies that \( v_{i,A^*_i} - p_{i,A^*_i} + e_i \geq v_{i,\mathcal{A}_i} - p_{i,\mathcal{A}_i} + e_i \) for any \( \mathcal{A} \in \mathcal{A} \), where the inequality is strict if \( \mathcal{A}_i \notin D_i(p; v) \). Since \( e_i \leq p_{i,\mathcal{A}_i} \), the left-hand term equals \( v_{i,A^*_i} - [\beta(e; p)]_{i,A^*_i} \). The right-hand term is also greater than \( v_{i,\mathcal{A}_i} - [\beta(e; p)]_{i,\mathcal{A}_i} \). For any \( i \in \mathcal{N} \), we obtain then: i) \( \mathcal{A}^*_i \in D_i(\beta(e; p); v) \), ii) \( \mathcal{A}_i \notin D_i(\beta(e; p); v) \) if \( \mathcal{A}_i \notin D_i(p; v) \) or equivalently \( D_i(\beta(e; p); v) \subseteq D_i(p; v) \). Since this last inclusion holds for any \( i \in \mathcal{N} \), then (1) \( \Rightarrow \mathcal{A}^* \in L^*(\beta(e; p)) \). On the whole, we have shown that \((\beta(e; p), \mathcal{A}^*) \) forms a quasi-CE. Furthermore, \( \mathcal{A}^*_i \in D_i(\beta(e; p); v) \cap D_i(p; v) \) and \( e_i \leq p_{i,\mathcal{A}^*_i} \) imply that \( \gamma^*_i(\beta(e; p)) = \gamma^*_i(p) + e_i \). In a second step and to conclude our proof, it is sufficient to show that for any \( \mathcal{A} \in \mathcal{A} \) such that \((p, \mathcal{A}) \) forms a quasi-CE, then \( e_i \leq p_{i,\mathcal{A}_i} \) for any \( i \in \mathcal{N} \) and \( \mathcal{A} \in \text{Arg max}_{\mathcal{A} \in \mathcal{A} | \mathcal{A}_i \in \{0, D_i(p; v)\}} \sum_{i \in \mathcal{N}} [\beta(e; p)]_{i,\mathcal{A}_i} \) (such that the above argument with \( \mathcal{A}^* := \mathcal{A} \) would apply to obtain that \((\beta(e; p), \mathcal{A}) \) forms a quasi-CE). If \((p, \mathcal{A}) \) forms a quasi-CE, then we have

\[
\sum_{i \in \mathcal{N}} (p_{i,\mathcal{A}_i} - e_i) \leq \sum_{i \in \mathcal{N}} [\beta(e; p)]_{i,\mathcal{A}_i} \leq \sum_{i \in \mathcal{N}} [\beta(e; p)]_{i,\mathcal{A}^*_i} = \sum_{i \in \mathcal{N}} (p_{i,\mathcal{A}^*_i} - e_i) = \sum_{i \in \mathcal{N}} (p_{i,\mathcal{A}_i} - e_i),
\]

where the second inequality comes from (1) and since \( \mathcal{A}_i \in D_i(p; v) \) for any \( i \), the first equality comes from \( e_i \leq p_{i,\mathcal{A}^*_i} \) for any \( i \) and the second equality comes from \( \mathcal{A}^*, \mathcal{A} \in L^*(p) \). On the whole we obtain that the inequalities in (5) stand as equalities and we have thus

\(^{15}\)With 3 valuations, it is possible by setting the price at the middle valuation since bidders are reporting their entire demand set.
$e_i \leq p_i_{\mathcal{A}_i}$ for any $i \in \mathcal{N}$ and $\mathcal{A} \in \operatorname{Arg}\max_{\mathcal{A} \in \mathcal{A}_i} \{0, D_i(p; v)\} \sum_{j \in \mathcal{N}} [\beta(e; p)]_{j, \mathcal{A}_j}$.

C Proof of Proposition 4.2

As argued before, it is sufficient to show that $\Gamma(p) = [\gamma^*(p)]^\dagger \cap CEP(\mathcal{N}, v)$ for any CE semi-truthful price vector $p$. The inclusion $\Gamma(p) \subseteq [\gamma^*(p)]^\dagger \cap CEP(\mathcal{N}, v)$ has already been established as a corollary of Proposition 4.1. We are thus left with the other inclusion. Consider $\tilde{\gamma} \in [\gamma^*(p)]^\dagger \cap CEP(\mathcal{N}, v)$ or equivalently $\tilde{\gamma} \in [\gamma^*(p)]^\dagger \cap \operatorname{Core}(\mathcal{N}, v)$. It is then sufficient to show that there exists $e \in \mathcal{E}(p)$ such that $\mathcal{P}(\tilde{\gamma}) = \beta(e; p)$.

By means of standard calculations, we have for any $\gamma_i \in \mathbb{R}^{N+1}_+$ with $\gamma_i \leq v_i\gamma$ (for any $i \in \mathcal{N}$):

$$\max_{\mathcal{A} \in \mathcal{A}_i} \sum_{j \in \mathcal{N}} [\mathcal{P}(\gamma)]_{j, \mathcal{A}_j} = \max_{\mathcal{A} \in \mathcal{A}_i} \sum_{j \in \mathcal{N}} \max \left\{ 0, v_j, \mathcal{A}_{j} - \gamma_j \right\} = \max_{\mathcal{A} \in \mathcal{A}_i} \sum_{j \in \mathcal{N}} v_j, \mathcal{A}_{j} - \gamma_j = \max_{\mathcal{A} \in \mathcal{A}_i} \sum_{j \in \mathcal{N}} \max \left\{ 0, v_j, \mathcal{A}_{j} - \gamma_j \right\} = \max_{\mathcal{A} \in \mathcal{A}_i} \sum_{j \in \mathcal{N}} \max \left\{ 0, v_j, \mathcal{A}_{j} - \gamma_j \right\}.$$  

Furthermore, if $\gamma \in \operatorname{Core}(\mathcal{N}, v)$, then $S = \mathcal{N}$ is a solution of the maximization program $\max_{\mathcal{A} \in \mathcal{A}_i} \sum_{j \in \mathcal{N}} [\mathcal{P}(\gamma)]_{j, \mathcal{A}_j}$ or equivalently from the calculation above there exists $\mathcal{A}^* \in \mathcal{A}^* \subseteq \mathcal{A}_i$ such that $\mathcal{P}(\mathcal{A}^*) \leq \mathcal{P}(\mathcal{A})$ for any $\mathcal{A} \in \mathcal{A}_i$ which is a solution of the maximization program $\max_{\mathcal{A} \in \mathcal{A}_i} \sum_{j \in \mathcal{N}} [\mathcal{P}(\gamma)]_{j, \mathcal{A}_j}$. Since $\tilde{\gamma} \in \operatorname{Core}(\mathcal{N}, v)$, take $\tilde{\mathcal{A}}^* = \tilde{\mathcal{A}}^* \subseteq \mathcal{A}^*$ such that $\mathcal{P}(\tilde{\gamma}) = \beta(e; p)$ (footnote 9). Since $\tilde{\gamma} \in [\gamma^*(p)]^\dagger$, we have also $\gamma_i \leq e_i \in \mathbb{R}^{N+1}_+$.

D Proof of Proposition 4.3

The discount rule defined by M&P (p. 348) for bidder $i$ at a quasi-CE price vector $p$ is given by $
abla_{i, I} (p) := \max_{\mathcal{A} \in \mathcal{A}_i} \sum_{j \in \mathcal{N}} p_j, \mathcal{A}_j - \max_{\mathcal{A} \in \mathcal{A}_i} \sum_{j \in \mathcal{N} \setminus \{i\}} p_j, \mathcal{A}_j$. We now show that $\nabla_{i, I} (p) = \nabla (p)$ for any $i \in \mathcal{N}$ if $p$ is a CE semi-truthful price vector. For $i \in \mathcal{N}$, let $e_i$ denote the vector in $\mathbb{R}^{N+1}_+$ such that $e_i = 0$ if $j \neq i$ and $e_i = 1$. For any scalar $\lambda \geq 0$ such that
\( \beta(\lambda^e; p) \) is a CE semi-truthful price vector, we have by means of standard calculations:

\[
\max_{A \in A|A_j \in \{\emptyset, D_j(p; v)\}} \sum_{j \in N} [\beta(\lambda^e; p)]_{j,A_j} \equiv \max_{A \in A} \sum_{j \in N} [\beta(\lambda^e; p)]_{j,A_j} \equiv \max_{S \subseteq N} \{ \max_{j \in S} w(S) - \sum_{j \in S} \gamma^*_j(\beta(\lambda^e; p)) \}
\]

\[
= \max\{ \max_{S \subseteq N, \lambda \in S} [w(S) - \sum_{j \in S} \gamma^*_j(p)], \max_{S \subseteq N, i \in S} [w(S) - \sum_{j \in S} \gamma^*_j(p)] - \lambda \}
\]

\[
eq \max_{\text{eq. (6)}} \{ \max_{A \in A} \sum_{j \in N \setminus \{i\}} p_{j,A_j} \} - \lambda \}
\]

where \( \gamma^*_j(\beta(\lambda^e; p)) \in \text{Core}(N, v) \) guarantees that \( S = N \) is a solution of the above maximization programs of the form \( \max_{S \subseteq N} \).

The positive scalar \( \tau_i(p) \) is defined as the greatest scalar \( \lambda \geq 0 \) such that there exists \( A^* \in A \) with \( A^* \in \text{Arg} \max_{A \in A|A_j \in \{\emptyset, D_j(p; v)\}} \sum_{j \in N} [\beta(\lambda^e; p)]_{j,A_j} \), \( A^* \in L^*(p) \) and \( A_j^* \in D_j(\beta(\lambda^e; p); v) \) for any \( j \). With the above calculation, \( \tau_i(p) \) corresponds also to the greatest scalar \( \lambda \geq 0 \) such that \( \sum_{j \in N} p_{j,A_j} - \lambda \geq \sum_{j \in N \setminus \{i\}} p_{j,A_j}, \) i.e. \( \tau_i(p) = \tau_i^{MP}(p) \).

From the definition of \( \tau_i(p) \), the payments after a maximal discount rule are smaller than after the MP discount rule. We now show that bidders’ payoffs after the MP discount rule are smaller than the Vickrey payoffs. Let \( A^* \) be an assignment such that \( (p, A^*) \) forms a quasi-CE and thus a CE since \( p \) is semi-truthful. The inequality \( \gamma^*_i(\delta_{MP}(p)) \leq \gamma^*_i(p) \) is equivalent to:

\[
\sum_{j \in N} p_{j,A_j^*} - p_{i,A_i^*} + \sum_{j \in N} p_{j,A_j^*} - \max_{A \in A} \{ \sum_{j \in N \setminus \{i\}} p_{j,A_j} \} \leq \sum_{j \in N} v_{j,A_j^*} - \max_{A \in A} \{ \sum_{j \in N \setminus \{i\}} v_{j,A_j} \}.
\]

Since \( A_j^* \in D_j(p; v) \) for any \( j \in N \setminus \{i\} \) this is also equivalent to \( \sum_{j \in N \setminus \{i\}} \gamma^*_i(p) + \max_{A \in A} \{ \sum_{j \in N \setminus \{i\}} p_{j,A_j} \} \geq \max_{A \in A} \{ \sum_{j \in N \setminus \{i\}} v_{j,A_j} \}. \) This last inequality holds since \( \gamma^*_j(p) + p_{j,A_j} \geq v_{j,A_j} \) for any \( j \in N \setminus \{i\} \) and \( A \in A \), which concludes the proof.